Description of the Radiation Field

Based on:

Chapter 1 of Rybicki & Lightman, *Radiative Processes in Astrophysics*,

and

Chapter 12 of Shu, *Radiation*.

1 Introduction

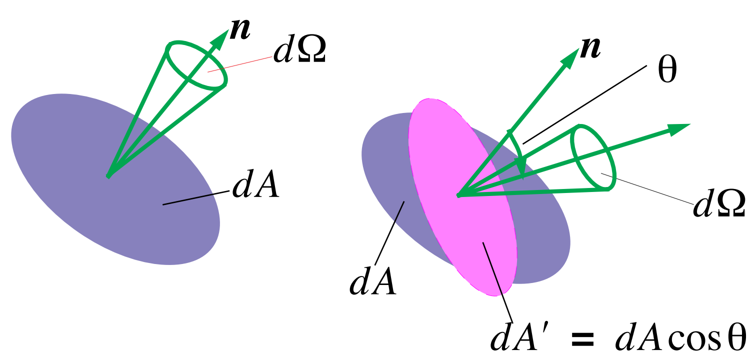
We see astrophysical objects in different wavebands from radio through to optical to X-ray and gamma-ray because of the radiation they emit, which propagates, both through the source and the intervening medium. Therefore, it is important to understand the properties of the radiation field and the manner in which it is described.

One of the most important diagnostics of radiation from an astrophysical source is that afforded by polarisation. For example, the direction of the magnetic field in a synchrotron emitting source is perpendicular to the direction of the

E-vector of the predominantly linear polarised radiation. Polarisation is also affected by the medium through which transverse waves travel (Faraday rotation). Hence it is important to have a sound theoretical basis from which to discuss polarisation.

2 Specific intensity and flux

2.1 Definition of specific intensity



Specific intensity is defined by:

Electromagnetic energy passing through surface normal to surface within elementary solid angle $= I_\nu d\nu dA dt d\Omega$ *(1)*

(Solid angle: $d\Omega$; Specific intensity: I_ν)

In terms of circular frequency ω:

Electromagnetic energy passing through surface normal to surface $= I_\omega d\omega dA dt d\Omega$ *(2)*

We often require the energy flowing through dA at an angle θ to the normal. We can derive the relevant expression in the following way. Consider the elementary surface dA' which is

- normal to the ray, and
- the projection of dA (see the above figure)

Then

$$dA' = dA\cos\theta \qquad (3)$$

and

Electromagnetic energy passing through surface at an angle θ to surface $= I_\nu(\theta, \phi)\cos\theta d\nu dA dt d\Omega$ *(4)*

We put

$$I_\nu = I_\nu(\theta, \phi) \qquad (5)$$

to emphasize the fact that the specific intensity can vary with direction with respect to the normal $\boldsymbol{n}$.

Units of I_ν:

We have

$$\text{Joules} = I_\nu \times \text{frequency} \times \text{area} \times \text{time} \times \text{solid angle} \qquad (6)$$

so that the units of I_ν are:

$$\text{Watts m}^{-2}\text{Hz}^{-1}\text{ Str}^{-1} \qquad (7)$$

or

$$\text{ergs s}^{-1}\text{cm}^{-2}\text{Hz}^{-1}\text{Sr}^{-1} \quad (8)$$

2.2 Flux density

The ***flux density***, F_ν, is the power per unit area of the radiation field. It is therefore defined by

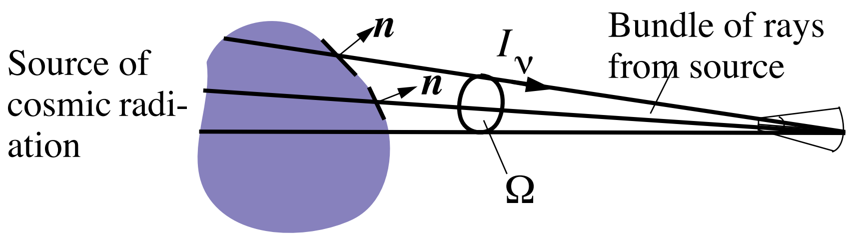
$$dF_\nu = I_\nu \cos\theta\, d\Omega \quad (9)$$

The flow of energy per unit area per unit time per unit frequency through a surface with normal $\boldsymbol{n}$ is given by:

$$F_\nu(\text{Wm}^{-2}\text{Hz}^{-1}) = \int_\Omega I_\nu \cos\theta\, d\Omega \quad (10)$$

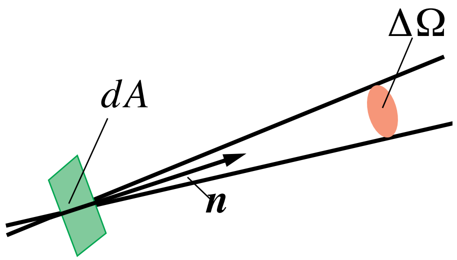
The density in this case refers to the "per unit frequency" part. More about this below.

2.3 Relationship of flux density to power at telescope



The solid angle of the bundle of rays is usually defined by the resolving power of the telescope. The power per unit area received by the telescope is $\int_\Omega I_\nu \cos\theta\, d\Omega$. Frequently in radio astronomy, you will hear people refer to the flux density of a source.

2.4 Surface brightness



For a small region of solid angle normal to the direction in which a telescope is pointing, we have:

$$\Delta F_\nu = I_\nu \Delta\Omega \Rightarrow I_\nu = \frac{\Delta F_\nu}{\Delta\Omega} \qquad (11)$$

Hence, the specific intensity is the flux received per unit of telescope area per unit of solid angle. For this reason, some astronomers, and particularly radio astronomers, refer to the specific intensity as ***surface brightness***.

This equation is also frequently used to estimate surface brightness of a source from an image when the image is represented in terms of ***flux density per beam.*** More about this later.

Units of flux density

Often, especially in radio astronomy, we use the unit of a Jansky (after one of the discoveries of cosmic radio radiation)

$$1 \text{ Jansky (Jy)} = 10^{-26}\ \text{W m}^{-2}\text{Hz}^{-1} \qquad (12)$$

2.5 Momentum flux density

Since the energy passing through a surface in a given direction is

$$dE_\nu = I_\nu \cos\theta\, d\nu\, dt\, d\Omega\, dA \qquad (13)$$

then the magnitude of momentum passing through the surface in the same direction is

$$dp_\nu = \frac{dE_\nu}{c} = \frac{I_\nu}{c}\cos\theta\, d\nu\, dt\, d\Omega\, dA \qquad (14)$$

However, momentum is a vector quantity. The *component* of momentum in the direction of the normal is

$$dp_\nu \cos\theta = \frac{I_\nu}{c}\cos^2\theta\, d\nu\, dt\, d\Omega\, dA \qquad (15)$$

The ***momentum flux density***, Π_ν, is the momentum per unit time per unit frequency per unit area passing in all directions through the surface, so that

$$\Pi_\nu = \frac{1}{c}\int_\Omega I_\nu \cos^2\theta \, d\Omega \qquad (16)$$

2.6 Integration over frequency

The total flux of energy per unit area between two frequencies ν_1 and ν_2 is just the integral of the flux density between these limits.

$$\text{Total flux} = F(\mathrm{Wm}^{-2}) = \int_{\nu_1}^{\nu_2} F_\nu d\nu \qquad (17)$$

Pressure is force per unit area, or equivalently, momentum flux density per unit area, so that the total radiation pressure on the surface can be found by integrating the momentum flux density over frequency.

$$\text{Pressure} = p(\mathrm{Nm}^{-2}) = \int_{\nu_1}^{\nu_2} \Pi_\nu d\nu \qquad (18)$$

The total intensity is just the integral of the specific intensity over frequency.

$$\text{Intensity} = I(\mathrm{Wm}^{-2}\mathrm{Str}^{-1}) = \int_{\nu_1}^{\nu_2} I_\nu d\nu \qquad (19)$$

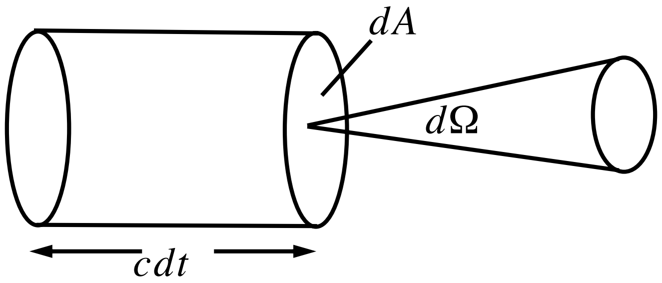
In particular the total flux is the frequency-integrated flux density. Sometimes people loosely refer to flux density as flux and I have seen pedantic thesis examiners insist on such carelessness being expunged from a thesis before it can be accepted.

2.7 Radiation energy density

Define

$$u_\nu(\Omega)\, dV d\Omega d\nu = \begin{matrix}\text{Energy in volume } dV \\ \text{solid angle } d\Omega \\ \text{and frequency interval } d\nu\end{matrix} \qquad (20)$$

Consider a cylinder of length cdt, cross-sectional area dA



The energy of radiation in cylinder within a cone of solid angle $d\Omega$ is:

$$\begin{aligned} dE &= u_\nu(\Omega)d\Omega dV d\nu = u_\nu(\Omega)d\Omega dA c dt d\nu \\ &= c u_\nu(\Omega) d\Omega dA dt d\nu \end{aligned} \tag{21}$$

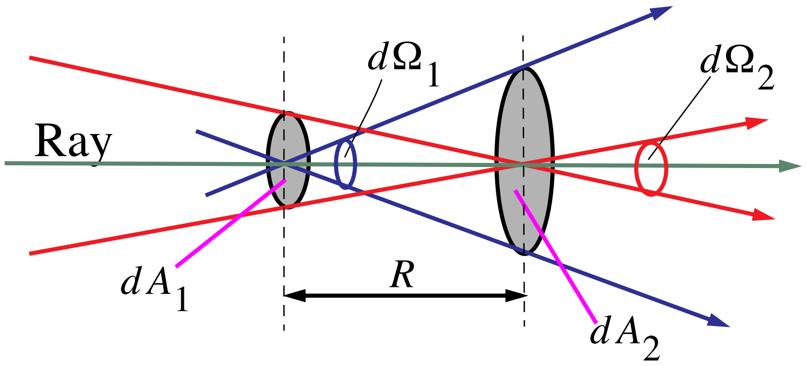
All of the radiation within the cylinder passes through dA in the time dt. Hence,

$$\begin{aligned} c u_\nu(\Omega) d\Omega dA dt d\nu &= I_\nu d\Omega dA dt d\nu \\ \Rightarrow c u_\nu(\Omega) &= I_\nu \\ u_\nu(\Omega) &= \frac{I_\nu}{c} \end{aligned} \tag{22}$$

Therefore, the total energy density at a point within the volume is given by:

$$\begin{aligned} u_\nu &= \frac{1}{c}\int_{4\pi} I_\nu d\Omega = \frac{4\pi}{c} J_\nu \\ \text{where} \qquad J_\nu &= \frac{1}{4\pi}\int_{4\pi} I_\nu d\Omega = \textbf{\textit{Mean intensity}} \end{aligned} \tag{23}$$

2.8 Constancy of specific intensity in free space



$$d\Omega_1 = \frac{dA_2}{R^2} \qquad d\Omega_2 = \frac{dA_1}{R^2} \tag{24}$$

Consider the energy flux through an elementary surface dA_1 within solid angle $d\Omega_1$. Consider all of the photons which pass through dA_1 in this direction which then pass through dA_2. We take the solid angle of these photons to be $d\Omega_2$. By conservation of energy the energy passing through both surfaces *within the corresponding solid angles* is identical. Hence,

$$dE = I_\nu^1 dA_1 dt d\Omega_1 d\nu_1 = I_\nu^2 dA_2 dt d\Omega_2 d\nu_2 \tag{25}$$

The solid angles are given by:

$$d\Omega_1 = \frac{dA_2}{R^2} \qquad d\Omega_2 = \frac{dA_1}{R^2} \tag{26}$$

This implies that:

$$d\Omega_1 dA_1 = d\Omega_2 dA_2 = \frac{dA_1 dA_2}{R^2} \tag{27}$$

Since $d\nu_1 = d\nu_2$, then

$$I_\nu^1 = I_\nu^2 = \text{constant along ray} \tag{28}$$

2.9 Spontaneous emission

Various emission processes along a ray contribute to the specific intensity. The emissivity, in principle, is angle-dependent, e.g. synchrotron emission depends upon the angle between the emission direction and the magnetic field. The emissivity is defined by:

Energy radiated from volume dV
in time dt into freqency interval $d\nu$ $= dE_\nu = j_\nu dV dt d\nu d\Omega$ (29)
into solid angle $d\Omega$

Emissivity

If the emissivity is isotropic, then

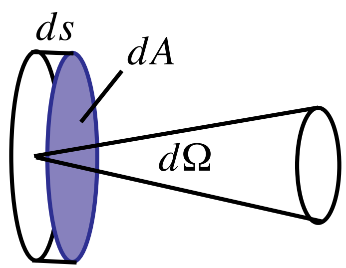
$$j_\nu = \frac{1}{4\pi} P_\nu \quad (30)$$

Radiated power per unit volume

The emission may be considered to be isotropic for two reasons:

1. The emission mechanism is independent of direction.
2. The emission may be considered to be the random superposition of a number of anisotropic emitters, e.g. synchrotron emission from a tangled magnetic field.

Effect on specific intensity



Energy added to beam from emission from within $dV = dAds$ is given by:

$$dE = j_\nu dV d\Omega dt d\nu = j_\nu dA ds d\Omega dt d\nu \quad (31)$$

This is radiated into the solid angle $d\Omega$ emerging from dA so that the change in specific intensity is given by:

$$dI_\nu dA d\Omega dt d\nu = j_\nu dA ds d\Omega dt d\nu$$

$$\Rightarrow \frac{dI_\nu}{ds} = j_\nu \quad (32)$$

2.10 Absorption

Often absorption is presented in the following form:

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu \quad (33)$$

Coefficient of absorption

and we shall see examples of this later on.

3 The radiative transfer equation

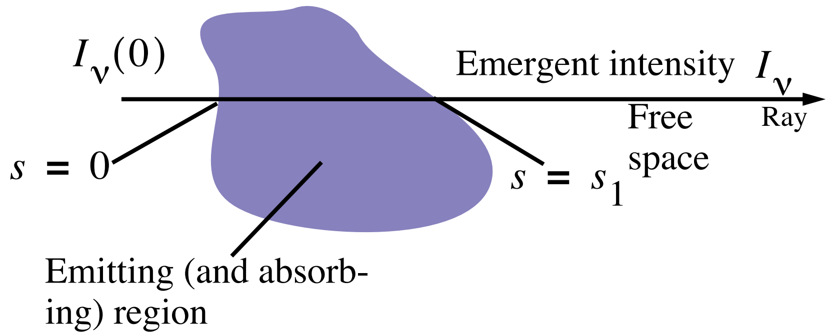
3.1 The fundamental equation

Putting emission and absorption into the one equation, we have

$$\frac{dI_\nu}{ds} = j_\nu - \alpha_\nu I_\nu \qquad (34)$$

Note that the emissivity can include scattering of photons from other directions into the direction being considered. This is what makes the solution of radiative transfer equations a challenging problem in general.

3.2 Solution for emission only (optically thin emission)



$$\frac{dI_\nu}{ds} = j_\nu \Rightarrow I_\nu(s) = I_\nu(0) + \int_0^s j_\nu(s')ds'$$
$$= I_\nu(0) + \int_0^{s_1} j_\nu(s')ds' \qquad (35)$$

In many cases:

- $I_\nu(0) = 0$
- We approximate the medium by one with constant properties. This gives,

$$I_\nu = j_\nu s_1 \qquad (36)$$

Once outside the emitting region, the specific intensity is constant, unless another emitting or absorbing region is encountered.

3.3 Absorption only

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu$$
$$\Rightarrow I_\nu(s) = I_\nu(s_0)\exp-\int_0^s \alpha_\nu(s')ds' \qquad (37)$$

This introduces the optical depth between 0 and s

$$\tau_\nu(s) = \int_0^s \alpha_\nu(s')ds' \quad (38)$$

In terms of τ_ν, the specific intensity along a ray is given by:

$$I_\nu(s) = I_\nu(0)e^{-\tau_\nu(s)} \quad (39)$$

3.4 Both emission and absorption

The differential form of the optical depth is

$$d\tau_\nu = \alpha_\nu ds \quad (40)$$

We can write the radiative transfer equation in the form:

$$\frac{dI_\nu}{\alpha_\nu ds} = S_\nu - I_\nu$$

$$\text{i.e.} \quad \frac{dI_\nu}{d\tau} = S_\nu - I_\nu \quad (41)$$

where

$$S_\nu = \frac{j_\nu}{\alpha_\nu} = \text{Source function} \quad (42)$$

Write the transfer equation as

$$\frac{dI_\nu}{d\tau_\nu} + I_\nu = S_\nu \quad (43)$$

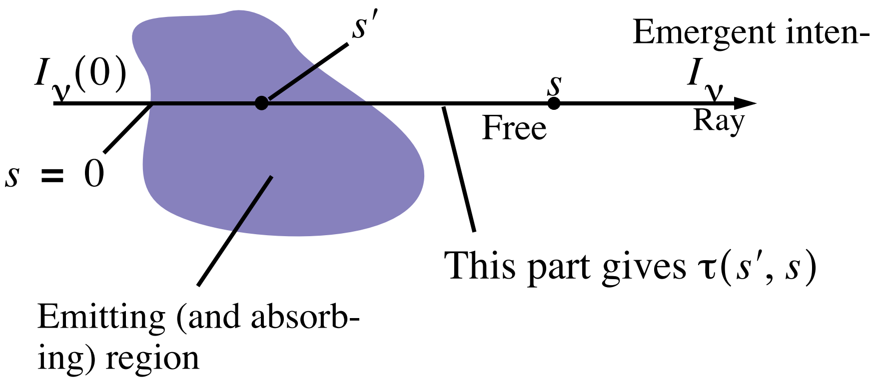
The integrating factor is e^{τ_ν} so that

$$\frac{d}{d\tau_\nu}\left[e^{\tau_\nu} I_\nu\right] = S_\nu e^{\tau_\nu} \Rightarrow e^{\tau_\nu} I_\nu = I_\nu(0) + \int_0^{\tau_\nu} e^{\tau_\nu'} S_\nu(\tau_\nu') d\tau_\nu \quad (44)$$

$$\Rightarrow I_\nu = e^{-\tau_\nu} I_\nu(0) + \int_0^{\tau_\nu} e^{-(\tau_\nu - \tau_\nu')} S_\nu(\tau_\nu') d\tau_\nu'$$

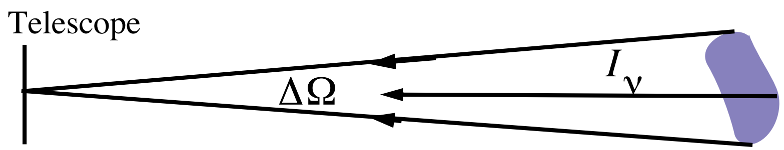
The optical depth delineates the region which contributes most significantly to the intensity of an emerging ray. To see this, consider

$$\begin{aligned} \tau_\nu - \tau_\nu' &= \int_0^s \alpha_\nu(s'')ds'' - \int_0^{s'} \alpha_\nu(s'')ds'' \\ &= \int_{s'}^s \alpha_\nu(s'')ds'' \\ &= \tau_\nu(s', s) \end{aligned} \quad (45)$$



It is obvious that the dominant contribution to the integral comes from regions wherein the optical depth, $\tau(s', s) < 1$.

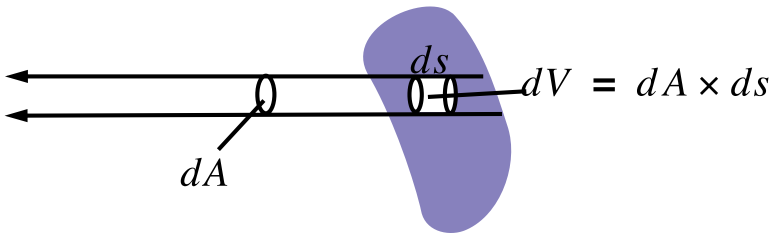
3.5 Relationship between flux and luminosity



The flux density received at the telescope is given by:

$$F_\nu = \int_{\Omega_{\text{source}}} I_\nu \cos\theta \, d\Omega \approx \int_{\Delta\Omega} I_\nu d\Omega \quad (46)$$

(We put $\theta \approx 0$ because for a distant source, all rays differ very little in their direction.)



Let dA be the cross-sectional area of the bundle of light rays at the source. For a distant source (distance R), the element of solid angle is given by

$$d\Omega = \frac{dA}{R^2} \quad (47)$$

where R is approximately the same for all parts of the source and dA is the cross-sectional area of a bundle of light rays as shown. Hence,

$$F_\nu \approx \frac{1}{R^2} \int_{\Delta\Omega} I_\nu dA \quad (48)$$

For optically thin radiation

$$I_\nu = \int_{\text{ray}} j_\nu ds$$

$$\Rightarrow F_\nu = \frac{1}{R^2} \int_{\text{Source}} j_\nu dA ds = \frac{1}{R^2} \int_{\text{Source}} j_\nu dV \quad (49)$$

where V is the volume (element dV).

The equation

$$F_\nu = \frac{1}{R^2}\int_{\text{Source}} j_\nu dV = \frac{1}{R^2} \times \text{Volume integrated emissivity} \qquad (50)$$

shows the origin of the inverse square law for flux density.

3.6 Isotropic emissivity

If the emissivity is isotropic

$$j_\nu = \frac{1}{4\pi}P_\nu = \frac{\text{Total power emitted per unit volume}}{4\pi \text{ solid angle}}$$
$$\Rightarrow F_\nu = \frac{1}{4\pi R^2}\int_{\text{Source}} P_\nu dV = \frac{L_\nu}{4\pi R^2} \qquad (51)$$

where

$$L_\nu = \text{Monochromatic luminosity}$$
$$= \text{Luminosity per unit frequency} \qquad (52)$$

3.7 Calculation of luminosity

Knowing the flux density, one can calculate the monochromatic luminosity, from

$$L_\nu = 4\pi R^2 F_\nu \qquad (53)$$

$$\text{Total luminosity} = L_{tot} = 4\pi R^2 \int_0^\infty F_\nu d\nu \qquad (54)$$

3.8 Example: The luminosity of a radio source

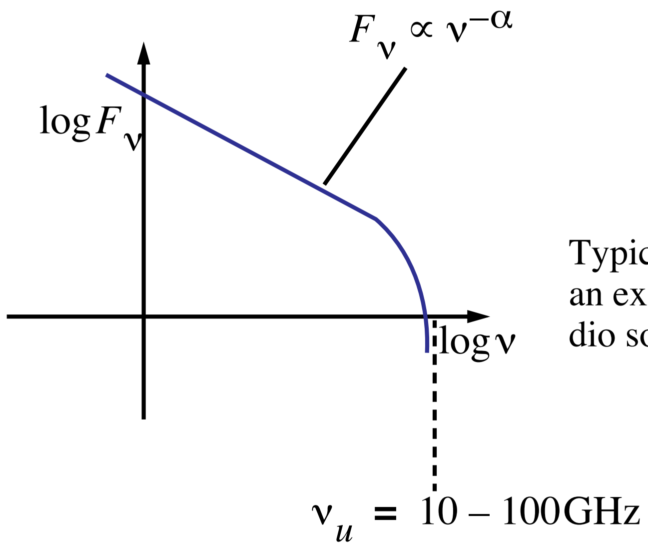
A typical extragalactic radio source would have a flux density at 1.4 GHz of a Jansky at a redshift of 0.1.

The distance (for small redshifts) is

$$D = \frac{cz}{H_0} = \frac{300,000 \times 0.1}{70}\ \text{Mpc}$$
$$\approx 430\,\text{Mpc} \qquad (55)$$

so that the monochromatic luminosity is

$$L_\nu = 4\pi \times (430 \times 3.1 \times 10^{22})^2 \times 10^{-26}\ \text{WHz}^{-1}$$
$$\approx 2.2 \times 10^{25}\,\text{WHz}^{-1} \qquad (56)$$



Typical spectrum of an extragalactic radio source

Typically such a source has a spectral index of $\alpha = 0.7$ between a generally undefined lower frequency, ν_l, and an upper cutoff frequency $\nu_u = 10 - 100\,\mathrm{GHz}$, so that the total luminosity,

$$
\begin{aligned}
L_{\mathrm{tot}} \approx \int_{\nu_l}^{\nu_u} L_{\nu_0}\left(\frac{\nu}{\nu_0}\right)^{-\alpha} d\nu &= L_{\nu_0}\nu_0 \int_{\nu_l/\nu_0}^{\nu_u/\nu_0} \left(\frac{\nu}{\nu_0}\right)^{-\alpha} d\left(\frac{\nu}{\nu_0}\right) \\
&= \frac{L_{\nu_0}\nu_0}{1-\alpha}\left[\left(\frac{\nu}{\nu_0}\right)^{1-\alpha}\right]_{\nu_l/\nu_0}^{\nu_u/\nu_0} \qquad (57) \\
&\approx \frac{L_{\nu_0}\nu_0}{1-\alpha}\left(\frac{\nu_u}{\nu_0}\right)^{1-\alpha}
\end{aligned}
$$

Thus, for the parameters of our source,

$$
\begin{aligned}
L_{\mathrm{tot}} &\approx \frac{2.2\times 10^{25} \times 1.4\times 10^{9}}{0.3} \times \left(\frac{10}{1.4}\right)^{0.3} \approx 2.7\times 10^{35}\,\mathrm{W} \qquad (58) \\
&\approx 7\times 10^{8} L_{\mathrm{sun}}
\end{aligned}
$$

4 Polarisation

4.1 Monochromatic plane wave

Plane wave solutions of Maxwell's equations:

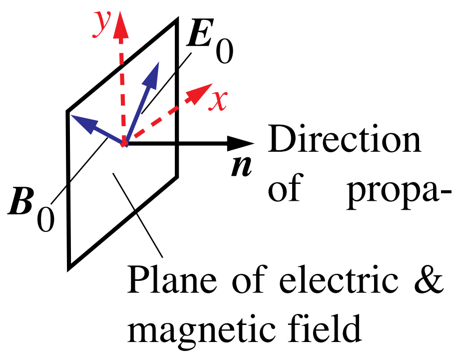
$$\boldsymbol{E} = \boldsymbol{E}_0 \exp i(\omega t - \boldsymbol{k}\cdot\boldsymbol{x}) \qquad \boldsymbol{B} = \boldsymbol{B}_0 \exp i(\omega t - \boldsymbol{k}\cdot\boldsymbol{x})$$

$$\omega = \text{circular frequency} = ck \qquad \boldsymbol{k} = \text{wave number} = k\boldsymbol{n}$$

$$\boldsymbol{E}_0 = \text{amplitude of Electric field} \tag{59}$$

$$\boldsymbol{B}_0 = \text{Amplitude of magnetic field} = \frac{k \times \boldsymbol{E}_0}{\omega} = c\boldsymbol{n} \times \boldsymbol{E}_0$$

$$\boldsymbol{B}_0 \cdot \boldsymbol{n} = \boldsymbol{E}_0 \cdot \boldsymbol{n} = 0$$



The electric vector determines all of the parameters of the wave. Since there are two independent components of $\boldsymbol{E}_0$ there are two modes of polarisation.

In general, we put

$$\boldsymbol{E}_0 = E_{0,1}\boldsymbol{e}_1 + E_{0,2}\boldsymbol{e}_2 \tag{60}$$

where $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are the unit vectors in the x and y directions.

Electromagnetic waves, far from the point of origin can be considered to be locally plane.

Electric vector at a point in space

Consider a wave at the location $\boldsymbol{x} = \boldsymbol{x}_0$. In component form the electric field of the wave may be written

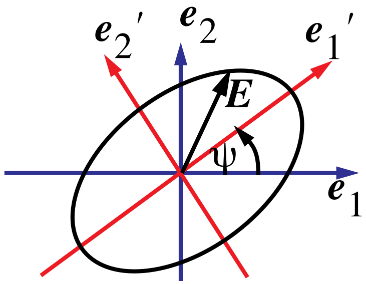
$$\begin{aligned} E_\alpha &= A_\alpha e^{i\delta_\alpha} e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{x}_0)} = A_\alpha e^{i\omega t} e^{i(\delta_\alpha - \boldsymbol{k}\cdot\boldsymbol{x}_0)} \\ &= A_\alpha e^{i(\omega t - \phi_\alpha)} \end{aligned} \tag{61}$$

Hence the real part of this wave may be expressed as

$$E_\alpha = A_\alpha \cos(\omega t - \phi_\alpha) \qquad \alpha = 1, 2 \tag{62}$$

The parameters ϕ_α are the phases of the two modes. They are not both arbitrary since the origin of time is arbitrary. However, the difference $\Delta\phi = \phi_2 - \phi_1$ is arbitrary.

4.2 The polarisation ellipse



Definition of axes for polarisation ellipse. ψ is the angle of rotation from the arbitrary axes to the principal axes.

Consider a general elliptically polarised monochromatic wave. The electric vector is given by:

$$\boldsymbol{E} = A_1 \cos(\omega t - \phi_1)\boldsymbol{e}_1 + A_2 \cos(\omega t - \phi_2)\boldsymbol{e}_2 \qquad (63)$$

N.B. The phases of both component are not free parameters since one phase can be adjusted by a change of the time origin. However, the relative phase $\phi_2 - \phi_1$ is arbitrary.

Write the electric field in axes corresponding to the principal axes of the ellipse:

$$\boldsymbol{E} = E_1 \cos\omega t\, \boldsymbol{e}_1' + E_2 \sin\omega t\, \boldsymbol{e}_2' \qquad (64)$$

The aim of the following is to determine the parameters E_1 and E_2 in terms of A_1, A_2, $\phi_2 - \phi_1$ and ψ. To do so, we use the relations between primed and unprimed unit vectors:

$$\begin{bmatrix} \boldsymbol{e}_1' \\ \boldsymbol{e}_2' \end{bmatrix} = \begin{bmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{bmatrix} \begin{bmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \end{bmatrix} \qquad (65)$$

Then write the electric field in the principal axis system in terms of the electric field in the arbitrary axes:

$$
\begin{aligned}
\boldsymbol{E} &= E_1 \cos\omega t[\cos\psi \boldsymbol{e}_1 + \sin\psi \boldsymbol{e}_2] \\
&\quad + E_2 \sin\omega t[-\sin\psi \boldsymbol{e}_1 + \cos\psi \boldsymbol{e}_2] \\
&= [E_1 \cos\psi \cos\omega t - E_2 \sin\psi \sin\omega t]\boldsymbol{e}_1 \\
&\quad + [E_1 \sin\psi \cos\omega t + E_2 \cos\psi \sin\omega t]\boldsymbol{e}_2 \\
&= A_1 \cos(\omega t - \phi_1)\boldsymbol{e}_1 + A_2 \cos(\omega t - \phi_2)\boldsymbol{e}_2 \\
&= [A_1 \cos\omega t \cos\phi_1 + A_1 \sin\omega t \sin\phi_1]\boldsymbol{e}_1 \\
&\quad + [A_2 \cos\omega t \cos\phi_2 + A_2 \sin\omega t \sin\phi_2]\boldsymbol{e}_2
\end{aligned} \tag{66}
$$

Equate the coefficients of $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\sin\omega t$, $\cos\omega t$ within those terms.

$$
\begin{aligned}
A_1 \cos\phi_1 &= E_1 \cos\psi \\
A_1 \sin\phi_1 &= -E_2 \sin\psi \\
A_2 \cos\phi_2 &= E_1 \sin\psi \\
A_2 \sin\phi_2 &= E_2 \cos\psi
\end{aligned} \tag{67}
$$

So far, so good, but this is not the best form in which to describe the relationship between these coefficients. The following quadratic relationships are easy to verify:

$$
\begin{aligned}
A_1^2 &= E_1^2 \cos^2\psi + E_2^2 \sin^2\psi \\
A_2^2 &= E_1^2 \sin^2\psi + E_2^2 \cos^2\psi
\end{aligned} \tag{68}
$$

together with:

$$
\begin{aligned}
A_1 A_2 \cos\phi_1 \cos\phi_2 &= E_1^2 \sin\psi \cos\psi \\
A_1 A_2 \sin\phi_1 \sin\phi_2 &= -E_2^2 \sin\psi \cos\psi \\
A_1 A_2 \sin\phi_1 \cos\phi_2 &= -E_1 E_2 \sin^2\psi \\
A_1 A_2 \cos\phi_1 \sin\phi_2 &= E_1 E_2 \cos^2\psi
\end{aligned} \tag{69}
$$

We now form the following combinations of the above:

$$A_1^2 + A_2^2 = E_1^2 + E_2^2$$
$$A_1^2 - A_2^2 = (E_1^2 - E_2^2)(\cos^2\psi - \sin^2\psi) \qquad (70)$$
$$= (E_1^2 - E_2^2)\cos 2\psi$$

$$A_1 A_2 \cos(\phi_2 - \phi_1) = (E_1^2 - E_2^2)\sin\psi\cos\psi$$
$$= \frac{(E_1^2 - E_2^2)}{2}\sin 2\psi \qquad (71)$$
$$A_1 A_2 \sin(\phi_2 - \phi_1) = E_1 E_2$$

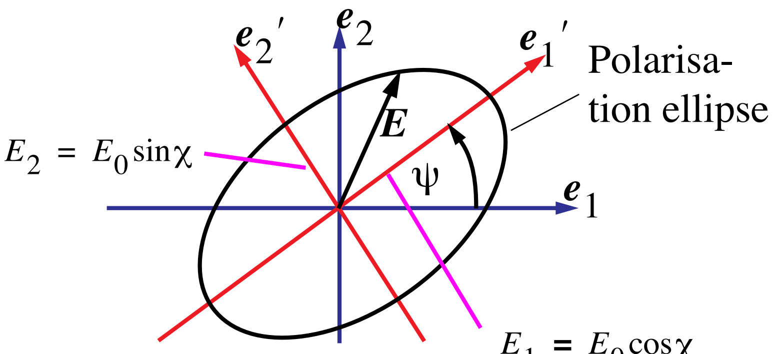
Now define the additional angle χ, the parameter t and the phase difference $\Delta\phi$ by:

$$E_1 = E_0\cos\chi \qquad E_2 = E_0\sin\chi$$
$$t = \frac{E_1}{E_2} = \cot\chi \qquad (72)$$
$$= \text{Ratio of semi-major to semi-minor axis}$$
$$\phi_2 - \phi_1 = \Delta\phi$$

The above quadratic relations become:

$$E_0^2 = A_1^2 + A_2^2$$
$$E_0^2\cos 2\chi\cos 2\psi = A_1^2 - A_2^2$$
$$E_0^2\cos 2\chi\sin 2\psi = 2A_1 A_2\cos\Delta\phi \qquad (73)$$
$$E_0^2\sin 2\chi = 2A_1 A_2\sin\Delta\phi$$

So the amplitudes A_1, A_2 and the phase difference $\Delta\phi$ define the parameters E_0, χ and ψ.



Relationship between parameters

4.3 Interdependence of parameters

Note that the 4 above equations are *not* independent. Take the last 3 equations; the sum of the squares of the left-hand sides is E_0^4. The sum of the squares of the right hand sides is

$$\begin{aligned}&(A_1^2 - A_2^2)^2 + 4A_1^2A_2^2(\cos^2\Delta\phi + \sin^2\Delta\phi)\\ &= (A_1^2 - A_2^2)^2 + 4A_1^2A_2^2\\ &= (A_1^2 + A_2^2)^2\end{aligned} \quad (74)$$

So the sum of the squares of these 3 equations gives:

$$\begin{aligned}E_0^4 &= (A_1^2 + A_2^2)^2\\ \Rightarrow E_0^2 &= A_1^2 + A_2^2\end{aligned} \quad (75)$$

which of course is the first equation.

5 The Stokes parameters – definitions

$$\begin{aligned}I &= \frac{c\varepsilon_0}{2}E_0^2 = \frac{c\varepsilon_0}{2}(A_1^2 + A_2^2)\\ Q &= \frac{c\varepsilon_0}{2}E_0^2\cos 2\chi\cos 2\psi = \frac{c\varepsilon_0}{2}(A_1^2 - A_2^2)\\ U &= \frac{c\varepsilon_0}{2}E_0^2\cos 2\chi\sin 2\psi = \frac{c\varepsilon_0}{2}(2A_1A_2\cos\Delta\phi)\\ V &= \frac{c\varepsilon_0}{2}E_0^2\sin 2\chi = \frac{c\varepsilon_0}{2}(2A_1A_2\sin\Delta\phi)\end{aligned} \quad (76)$$

(The reason for the factor of $(c\varepsilon_0)/2$ is the relation to the Poynting flux in the following.)

These equations can also be expressed in the form:

$$\begin{aligned} Q &= I\cos 2\chi \cos 2\psi \\ U &= I\cos 2\chi \sin 2\psi \\ V &= I\sin 2\chi \end{aligned} \tag{77}$$

Squaring each equation and adding:

$$I^2 = Q^2 + U^2 + V^2 \tag{78}$$

Note the close correspondence between these equations for $(I, 2\chi, 2\psi)$ and defining these angles and polar coordinates. This correspondence is exploited when we discuss the Poincare sphere.

5.1 Fractional polarisation

We define the ***fractional polarisation*** of a wave by:

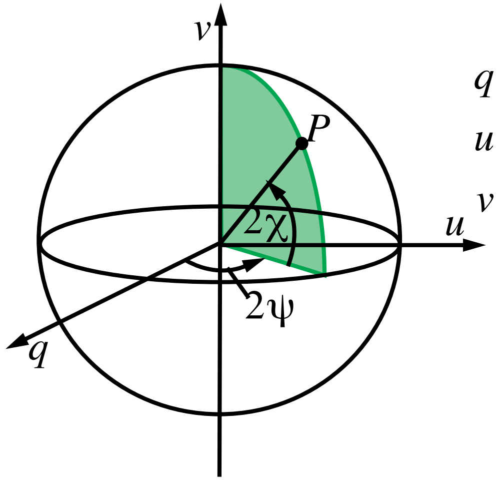
$$\begin{aligned} q &= \frac{Q}{I} = \cos 2\chi \cos 2\psi \\ u &= \frac{U}{I} = \cos 2\chi \sin 2\psi \\ v &= \frac{V}{I} = \sin 2\chi \end{aligned} \tag{79}$$

The direction of the principal axis is therefore given by:

$$\tan 2\psi = \frac{u}{q} = \frac{U}{Q} \tag{80}$$

5.2 The Poincare sphere

Polarised light can be represented in terms of the Poincare



$$q = \cos 2\chi \cos 2\psi$$
$$u = \cos 2\chi \sin 2\psi \quad (81)$$
$$v = \sin 2\chi$$

sphere. The Stokes parameters for fractional polarisation can be represented in terms of the parameters 2ψ and 2χ as polar angles.

The Poincare sphere makes it easy to determine the relevant ranges of ψ and χ. From the diagram it is obvious that

$$\begin{array}{ccc} 0 < 2\psi < 2\pi & & 0 < \psi < \pi \\ -\frac{\pi}{2} < 2\chi < \frac{\pi}{2} & \Rightarrow & -\frac{\pi}{4} < \chi < \frac{\pi}{4} \end{array} \quad (82)$$

Physically, the reason for this is as follows:

1.Rotation of an ellipse by ψ and $\psi + \pi$ give the same

ellipse.

2. Recall the definition of χ

$$E_1 = E_0 \cos\chi \qquad E_2 = E_0 \sin\chi$$
$$t = \frac{E_1}{E_2} = \cot\chi \qquad t^{-1} = \frac{E_2}{E_1} = \tan\chi \quad (83)$$

When χ varies between $\pm\pi/4$, E_2 varies between $\pm E_1$. This is the appropriate range for the semi-minor axis.

5.3 Linear and circular polarisation

5.3.1 Linear polarisation

If the phase difference between the two components in the arbitrary reference system is $\Delta\phi = 0$, then

$$V = c\varepsilon_0 A_1 A_2 \sin\Delta\phi = 0 \Rightarrow \sin 2\chi = 0$$
$$\Rightarrow 2\chi = 0, \pi \quad (84)$$

The only value of χ in the appropriate range is $\chi = 0$.

This implies that

$$E_1 = E_0 \cos\chi = E_0 \qquad E_2 = E_0 \sin\chi = 0 \quad (85)$$

Hence the electric field is:

$$\boldsymbol{E} = E_0 \cos\omega t \boldsymbol{e}_1' \qquad (86)$$

i.e. the electric vector oscillates in one direction – hence the name linear polarisation.

5.3.2 Circular polarisation

A purely ***circularly polarised*** wave is defined by equal amplitudes of the two components, differing in phase by $\pi/2$, i.e.

$$A_1 = \pm A_2 \qquad \Delta\phi = \pm\pi/2 \qquad (87)$$

Since,

$$Q = \frac{c\varepsilon_0}{2}(A_1^2 - A_2^2)$$
$$U = \frac{c\varepsilon_0}{2}(2A_1A_2\cos\Delta\phi) \qquad (88)$$

then

$$q = Q = 0$$
$$u = U = 0 \qquad (89)$$

The only remaining Stokes parameter in this case is:

$$V = \frac{c\varepsilon_0}{2}(2A_1A_2\cos\Delta\phi)$$
$$\Rightarrow v = \frac{\frac{c\varepsilon_0}{2}(2A_1A_2\sin\Delta\phi)}{\frac{c\varepsilon_0}{2}(A_1^2 + A_2^2)} = \pm 1 \qquad (90)$$

The equations defining ψ and ϕ are:

$$\cos 2\chi \cos 2\psi = 0$$
$$\cos 2\chi \sin 2\psi = 0 \qquad (91)$$
$$\sin 2\chi = \pm 1$$

The solution for this is

$$2\chi = \pm\frac{\pi}{2} \Rightarrow \chi = \pm\frac{\pi}{4} \qquad (92)$$

and

$$\psi = \text{arbitrary} \qquad (93)$$

Hence

$$E_2 = \pm E_1 \qquad (94)$$

and the two waves are:

$$\begin{aligned} \boldsymbol{E} &= E_0 \cos\omega t \boldsymbol{e}_1' + E_0 \sin\omega t \boldsymbol{e}_2' \\ \boldsymbol{E} &= E_0 \cos\omega t \boldsymbol{e}_1' - E_0 \sin\omega t \boldsymbol{e}_2' \end{aligned} \qquad (95)$$

The first solution represents a vector moving anti-clockwise in a circle as seen by an observer facing the wave – This is known as *left circularly polarised* or *positive helicity*.

The second represents a vector moving clockwise in a circle as seen by an observer facing the wave – This is known as *right circularly polarised* or *negative helicity*.

5.3.3 General elliptical polarisation

In the general case when

$$q, u, v \neq 0 \qquad (96)$$

$$\boldsymbol{E} = E_0 \cos\chi \cos\omega t \boldsymbol{e}_1' + E_0 \sin\chi \sin\omega t \boldsymbol{e}_2' \qquad (97)$$

When $v > 0$ and consequently $\chi > 0$ then $\boldsymbol{E}$ rotates anti-clockwise (since $\cos\chi > 0$ and $\sin\chi > 0$) and the wave is ***left-polarised***.

When $v < 0$ and consequently $\chi < 0$, $\boldsymbol{E}$ rotates clockwise (since $\cos\chi > 0$ and $\sin\chi < 0$) and the wave is ***right-polarised***.

5.3.4 Direction of the major axis

Take

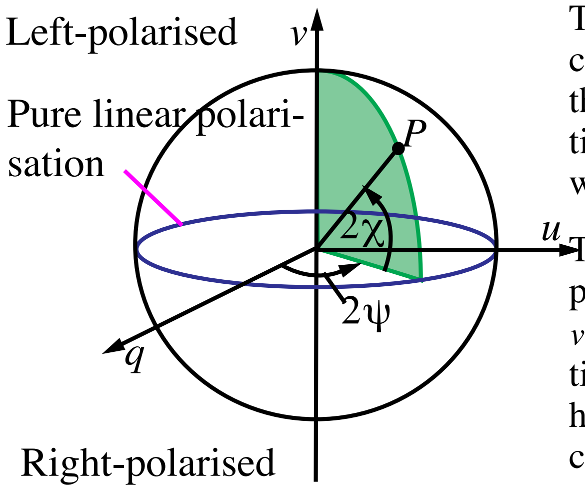
$$\begin{aligned} q &= \cos 2\chi \cos 2\psi \\ u &= \cos 2\chi \sin 2\psi \\ v &= \sin 2\chi \end{aligned} \qquad (98)$$

then it is clear that

$$\tan 2\psi = \frac{u}{q} \Rightarrow \psi = \frac{1}{2}\tan^{-1}\frac{u}{q} \qquad (99)$$

In the case of linear polarisation this is the direction of the line of oscillation of the electric field.

5.4 The Poincare sphere revisited



The point P on the Poincare sphere represents the fractional polarisation of a monochromatic wave.

The North and South poles of the sphere, $v = 1$ and $v = -1$ respectively, represent left-handed and right-handed circular polarisation.

5.5 Relationship to the Poynting flux

The Poynting flux is

$$\boldsymbol{S} = \frac{\boldsymbol{E} \times \boldsymbol{B}}{\mu_0} \tag{100}$$

For a transverse wave with wave-vector $\boldsymbol{k}$ and normal $\boldsymbol{n} = \dfrac{\boldsymbol{k}}{k}$:

$$\boldsymbol{B} = \frac{\boldsymbol{k} \times \boldsymbol{E}}{\omega} \Rightarrow \frac{\boldsymbol{E} \times \boldsymbol{B}}{\mu_0} = \frac{E^2}{\mu_0 \omega}\boldsymbol{n} \tag{101}$$

With $\omega = ck$,

$$\boldsymbol{S} = \frac{E^2}{\mu_0 c}\boldsymbol{n} = c\varepsilon_0 E^2 \boldsymbol{n} = c\varepsilon_0 [E_1^2 \cos^2 \omega t + E_2^2 \sin^2 \omega t]\boldsymbol{n} \tag{102}$$

Averaged over a period, the Poynting flux is:

$$\langle \boldsymbol{S} \rangle = c\varepsilon_0 \left(\frac{E_1^2}{2} + \frac{E_2^2}{2} \right) = \frac{c\varepsilon_0 E_0^2}{2} \tag{103}$$

The Stokes parameter I is the Poynting flux of electromagnetic energy.

6 Polarisation of a quasi-monochromatic wave

6.1 The electric field

As before, we write the electric field as

$$\begin{aligned} \boldsymbol{E} &= A_1 \cos(\omega t - \phi_1)\boldsymbol{e}_1 + A_2 \cos(\omega t - \phi_2)\boldsymbol{e}_2 \\ &= Re\left[A_1(t) e^{-i\phi_1(t)}\right]\boldsymbol{e}_1 + Re\left[A_2(t) e^{-i\phi_2(t)}\right]\boldsymbol{e}_2 \end{aligned} \tag{104}$$

The previous section is concerned with the case in which the waves are purely monochromatic so that A_α and ϕ_α are constant. In the following a complex notation based on the above is used.

We consider ***quasi-monochromatic*** waves for which

$$E_1(t) = A_1(t)e^{-i\phi_1(t)} \qquad E_2(t) = A_2(t)e^{-i\phi_2(t)} \tag{105}$$

where the time scale of variation of the waves is much longer than the wave period. This is relevant to the situation where the estimate of the Stokes parameters involves averages over many periods. For example, to consider a ra-

dio wave as a monochromatic wave, one would have to sample at the rate of once every 10^{-9} s or so. In reality, measurements at a radio telescope require integration times of about 5 minutes.

6.2 Stokes parameters for quasi-monochromatic waves

We define the Stokes parameters as time averages ($\langle\ \rangle$):

$$\begin{aligned}
I &= \frac{c\varepsilon_0}{2}\langle A_1^2 + A_2^2\rangle = \frac{c\varepsilon_0}{2}[\langle E_1 E_1^*\rangle + \langle E_2 E_2^*\rangle] \\
Q &= \frac{c\varepsilon_0}{2}\langle A_1^2 - A_2^2\rangle = \frac{c\varepsilon_0}{2}[\langle E_1 E_1^*\rangle - \langle E_2 E_2^*\rangle] \\
U &= c\varepsilon_0\langle A_1 A_2 \cos\Delta\phi\rangle = \frac{c\varepsilon_0}{2}[\langle E_1^* E_2\rangle + \langle E_1 E_2^*\rangle] \\
V &= c\varepsilon_0 A_1 A_2 \sin\Delta\phi = \frac{c\varepsilon_0}{2i}[\langle E_1^* E_2\rangle - \langle E_1 E_2^*\rangle]
\end{aligned} \tag{106}$$

In this case $I^2 \geq Q^2 + U^2 + V^2$, as we now show. Denote $C = \frac{c\varepsilon_0}{2}$, then

$$\begin{aligned} I^2 &= C^2[\langle E_1 E_1^* \rangle^2 + \langle E_2 E_2^* \rangle^2 + 2\langle E_1 E_1^* \rangle \langle E_2 E_2^* \rangle] \\ Q^2 &= C^2[\langle E_1 E_1^* \rangle^2 + \langle E_2 E_2^* \rangle^2 - 2\langle E_1 E_1^* \rangle \langle E_2 E_2^* \rangle] \\ U^2 &= C^2[\langle E_1 E_2^* \rangle^2 + \langle E_2 E_1^* \rangle^2 + 2\langle E_1 E_2^* \rangle \langle E_2 E_1^* \rangle] \\ V^2 &= C^2[-\langle E_1 E_2^* \rangle^2 - \langle E_2 E_1^* \rangle^2 + 2\langle E_1 E_2^* \rangle \langle E_2 E_1^* \rangle] \end{aligned} \tag{107}$$

These imply that

$$\begin{aligned} I^2 - (Q^2 + U^2 + V^2) = 4C^2[&\langle E_1 E_1^* \rangle \langle E_2 E_2^* \rangle \\ &- \langle E_1 E_2^* \rangle \langle E_2 E_1^* \rangle] \end{aligned} \tag{108}$$

Since we dealing with time averages, e.g.

$$\langle E_1 E_1^* \rangle = \frac{1}{T}\int_0^T \langle E_1(t) E_1^*(t) \rangle dt \tag{109}$$

where T is the time of integration, then, using the Cauchy-Schwarz inequality,

$$\langle E_1 E_1^* \rangle \langle E_2 E_2^* \rangle \geq \langle E_1 E_2^* \rangle \langle E_2 E_1^* \rangle \tag{110}$$

implying that

$$I^2 - (Q^2 + U^2 + V^2) \geq 0 \tag{111}$$

7 Superposition of independent waves

Another important case to consider is where the radiation received by a detector is composed of a number of independent components - "independent" meaning that the amplitudes and phases of the components are uncorrelated. We put

$$
\begin{aligned}
E_\alpha &= \sum_i E^i_\alpha \qquad \alpha = 1, 2 \\
E^i_\alpha &= A^i_\alpha e^{-i\phi^i_\alpha} \\
A^i_\alpha &= \text{Amplitude of } \alpha \text{ part of } i^{\text{th}} \text{ component} \\
\phi^i_\alpha &= \text{Phase of } \alpha \text{ part of } i^{\text{th}} \text{ component}
\end{aligned}
\qquad (112)
$$

The Stokes parameters consist of terms of the form $\langle E_\alpha E^*_\beta \rangle$ which we can write as:

$$
\langle E_\alpha E^*_\beta \rangle = \sum_i \sum_j \langle E^{(i)}_\alpha E^{*(j)}_\beta \rangle \qquad (113)
$$

Now,

$$
\begin{aligned}
E^i_\alpha E^{*j}_\beta &= A^i_\alpha A^j_\beta e^{-i(\phi^i_\alpha - \phi^j_\beta)} = A^i_\alpha A^j_\beta e^{-i\Delta\phi^{ij}_{\alpha\beta}} \\
\Delta\phi^{ij}_{\alpha\beta} &= \text{Phase difference between the } \alpha \text{ and } \beta \text{ parts of components } i \text{ and } j
\end{aligned}
\qquad (114)
$$

The essence of independent waves is that the phase differences between them be randomly distributed over $[0, 2\pi]$. For this reason,

$$
\langle E^i_\alpha E^{*j}_\beta \rangle = 0 \text{ when } i \neq j \qquad (115)
$$

Hence,

$$
\langle E^i_\alpha E^{*j}_\beta \rangle = \sum_i \langle E^i_\alpha E^{*i}_\beta \rangle \qquad (116)
$$

and

$$
\begin{aligned}
I &= \sum_i I^i \qquad Q = \sum_i Q^i \\
U &= \sum_i U^i \qquad V = \sum_i V^i
\end{aligned}
\qquad (117)
$$

i.e. the Stokes parameters are the sums of the Stokes parameters of the individual waves.

8 Partially polarised radiation

8.1 Separation into polarised and unpolarised components

Consider the relations for the Stokes parameters for an arbitrary quasi-monochromatic wave:

$$\begin{aligned} I &= C[\langle E_1 E_1^* \rangle + \langle E_2 E_2^* \rangle] \\ Q &= C[\langle E_1 E_1^* \rangle - \langle E_2 E_2^* \rangle] \\ U &= C[\langle E_1 E_2^* \rangle + \langle E_2 E_1^* \rangle] \\ V &= \frac{C}{i}[\langle E_1 E_2^* \rangle - \langle E_2 E_1^* \rangle] \end{aligned} \tag{118}$$

If, on average, the amplitudes of the two parts of the wave are the same, then

$$Q = \langle E_1 E_1^* \rangle - \langle E_2 E_2^* \rangle = 0 \tag{119}$$

Consider

$$U = \langle E_1 E_2^* \rangle + \langle E_2 E_1^* \rangle = \langle 2 A_1 A_2 \cos \Delta\phi \rangle \tag{120}$$

If the phase difference between the two components varies in such a way, that it averages to zero, then $U = 0$. Similarly for V. Radiation with these properties is called *unpolarised* and is distinguished by:

$$I \neq 0 \qquad Q = U = V = 0 \tag{121}$$

Similarly, if radiation is composed of a number of independent components and the phase difference $\Delta\phi_{12}^{ii}$ is randomly distributed, then we also have

$$Q = U = V = 0. \tag{122}$$

We separate EM radiation into polarised and unpolarised components as follows:

$$\begin{bmatrix} I \\ Q \\ U \\ V \end{bmatrix} = \begin{bmatrix} I-(Q^2+U^2+V^2)^{1/2} \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} (Q^2+U^2+V^2)^{1/2} \\ Q \\ U \\ V \end{bmatrix} \qquad (123)$$

Unpolar-
ised compo-

Polarised
component

The fractional polarisation is:

$$r = \frac{I_{\text{pol}}}{I} = \frac{(Q^2+U^2+V^2)^{1/2}}{I} \qquad (124)$$

8.2 Polarisation from astrophysical sources

In general, radiation from astrophysical sources is only weakly polarised – at the level of 1 or 2%. However, radiation from synchrotron sources can be very highly polarised – up to 50-70% in some cases and this is often a good indication of the presence of synchrotron emission.

Nevertheless, even polarisation at the level of 1 or 2% can be extremely important.