Relativistic Effects

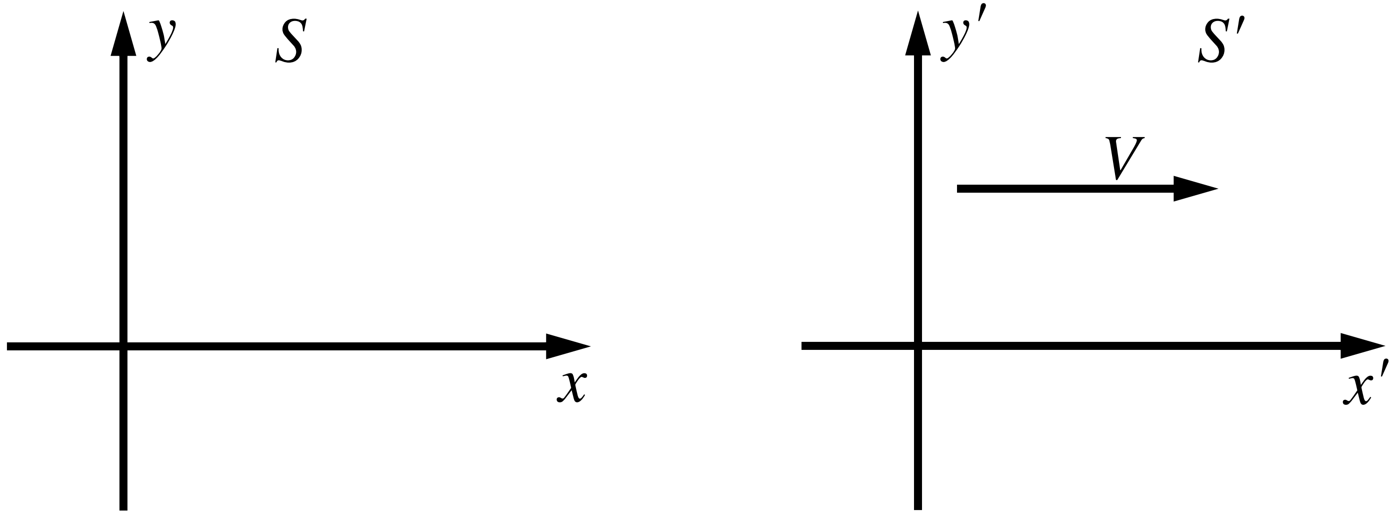
1 Introduction

The radio-emitting plasma in AGN contains electrons with relativistic energies. The Lorentz factors of the emitting electrons are of order 10^{2-6}. We now know that the bulk motion of the plasma is also moving relativistically – at least in some regions although probably "only" with Lorentz factors about 10 or so. However, this has an important effect on the properties of the emitted radiation – principally through the effects of relativistic beaming and doppler shifts in frequency. This in turn affects the inferred parameters of the plasma.

2 Summary of special relativity

For a more complete summary of 4-vectors and Special Relativity, see Rybicki and Lightman, *Radiative Processes in Astrophysics*, or Rindler, *Special Relativity*

2.1 The Lorentz transformation



The primed frame is moving wrt to the unprimed frame with a velocity v in the x–direction. The coordinates in the primed frame are related to those in the unprimed frame by:

$$x' = \Gamma(x - vt) \qquad t' = \Gamma\left(t - \frac{Vx}{c^2}\right)$$

$$y' = y \qquad z' = z \qquad (1)$$

$$\beta = \frac{v}{c} \qquad \Gamma = \frac{1}{\sqrt{1-\beta^2}}$$

We put the space-time coordinates on an equal footing by putting $x^0 = ct$. The the $x - t$ part of the Lorentz transformation can be written:

$$x' = \Gamma(x - \beta x^0) \qquad x^{0\prime} = \Gamma(x^0 - \beta x) \qquad (2)$$

The reverse transformation is:

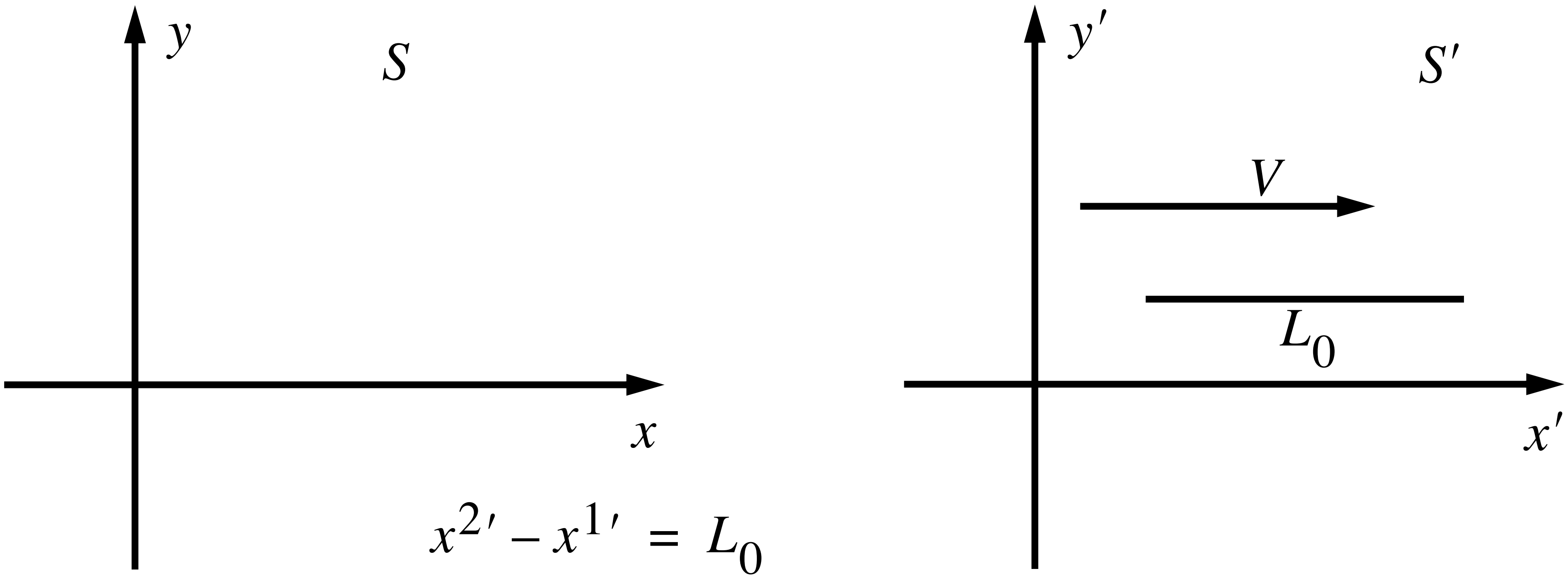
$$x = \Gamma(x' + Vt') \qquad y = y' \qquad z = z'$$

$$t = \Gamma\left(t' + \frac{Vx'}{c^2}\right) \tag{3}$$

i.e.,

$$x = \Gamma(x' + \beta x^{0\prime}) \qquad x^0 = \Gamma(x^{0\prime} + \beta x') \tag{4}$$

2.2 Lorentz–Fitzgerald contraction

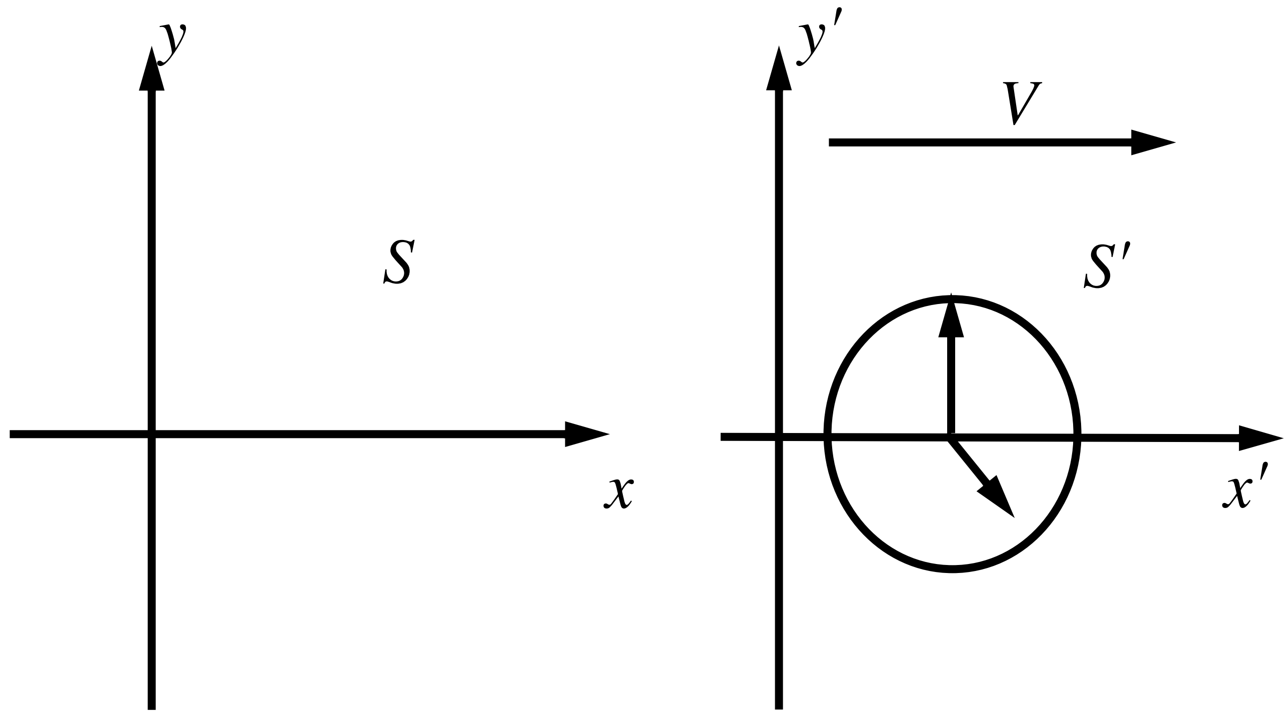


$$x^{2\prime} - x^{1\prime} = L_0$$

$$x^{2\prime} - x^{1\prime} = \Gamma[(x_2 - x_1) - V(t_2 - t_1)]$$

$$L_0 = \Gamma L \Rightarrow L = \Gamma^{-1} L_0$$

2.3 Time dilation

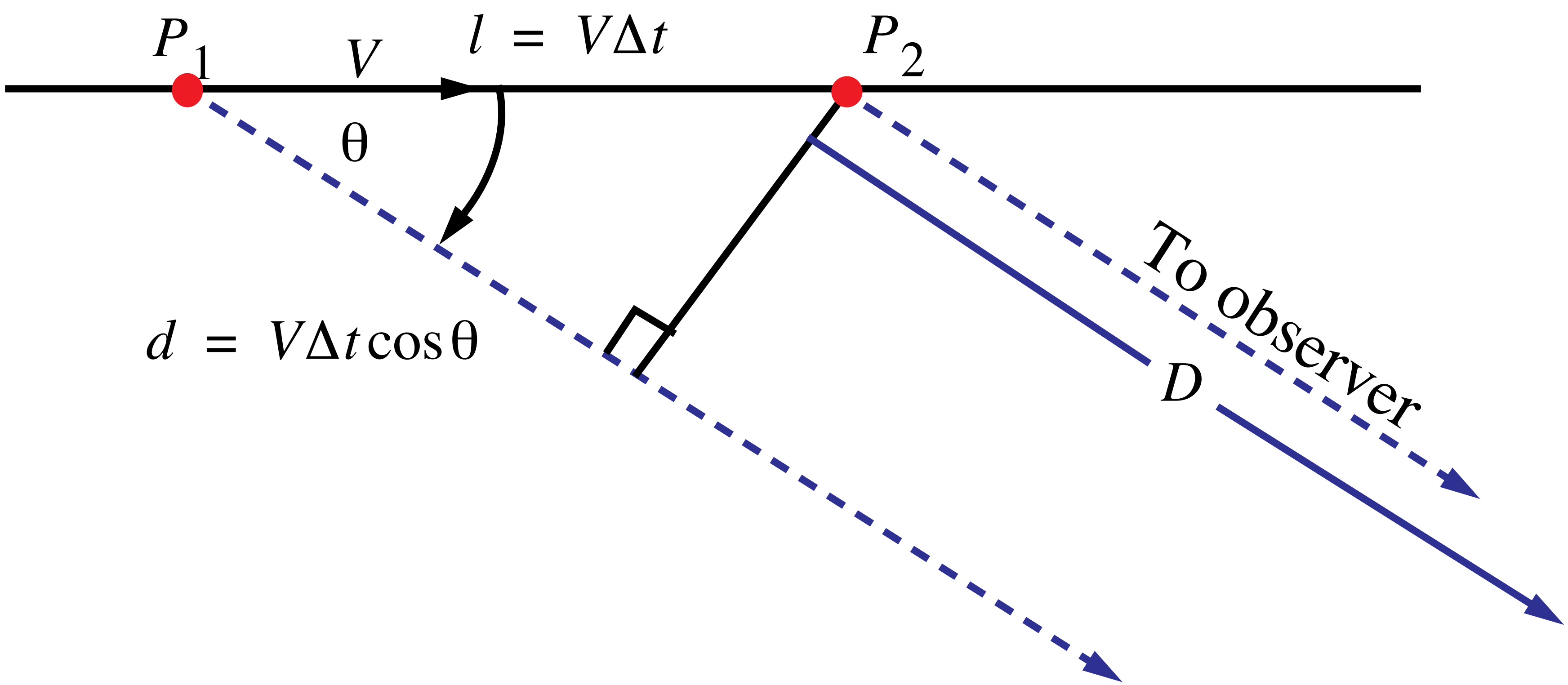


Consider a clock at a stationary position in the moving frame which registers a time interval T_0. The corresponding time interval in the “lab” frame is given by:

$$\begin{aligned} T &= t_2 - t_1 = \Gamma[(t_2' - t_1') - V(x_2' - x_1')/c^2] \\ &= \Gamma(t_2' - t_1') = \Gamma T_0 \end{aligned} \qquad (5)$$

i.e. the clock appears to have slowed don by a factor of Γ

2.4 Doppler effect



The Doppler effect is very important when describing the effects of relativistic motion in astrophysics. The effect is the combination of both relativistic time dilation and time retardation. Consider a source of radiation which emits one period of radiation over the time Δt it takes to move from P_1 to P_2. If ω_{em} is the emitted circular frequency of the radiation in the rest frame, then

$$\Delta t' = \frac{2\pi}{\omega_{em}} \tag{6}$$

and the time between the two events in the observer's frame is:

$$\Delta t = \Gamma \Delta t' = \Gamma \frac{2\pi}{\omega_{em}} \qquad (7)$$

However, this is not the observed time between the events because there is a time difference involved in radiation emitted from P_1 and P_2. Let

$$D = \text{distance to observer from } P_2 \qquad (8)$$

and

$$\begin{aligned} t_1 &= \text{time of emission of radiation from } P_1 \\ t_2 &= \text{time of emission of radiation from } P_2 \end{aligned} \qquad (9)$$

Then, the times of reception, $t_1^{\rm rec}$ and $t_2^{\rm rec}$ are:

$$
\begin{aligned}
t_1^{\rm rec} &= t_1 + \frac{D + V\Delta t\cos\theta}{c} \\
t_2^{\rm rec} &= t_2 + \frac{D}{c}
\end{aligned}
\tag{10}
$$

Hence the period of the pulse received in the observer's frame is

$$
\begin{aligned}
t_2^{\mathrm{rec}} - t_1^{\mathrm{rec}} &= \left(t_2 + \frac{D}{c}\right) - \left(t_1 + \frac{D + V\Delta t\cos\theta}{c}\right) \\
&= (t_2 - t_1) - \frac{V}{c}\Delta t\cos\theta \qquad (11) \\
&= \Delta t\left(1 - \frac{V}{c}\cos\theta\right)
\end{aligned}
$$

Therefore,

$$\frac{2\pi}{\omega_{obs}} = \Gamma \frac{2\pi}{\omega_{em}} \left(1 - \frac{V}{c}\cos\theta\right)$$

$$\Rightarrow \omega_{obs} = \frac{\omega_{em}}{\Gamma\left(1 - \frac{V}{c}\cos\theta\right)} = \frac{\omega_{em}}{\Gamma(1 - \beta\cos\theta)} \qquad (12)$$

The factor Γ is a pure relativistic effect, the factor $(1 - \beta\cos\theta)$ is the result of time retardation. In terms of linear frequency:

$$\nu_{obs} = \frac{\nu_{em}}{\Gamma(1 - \beta\cos\theta)} \qquad (13)$$

The factor

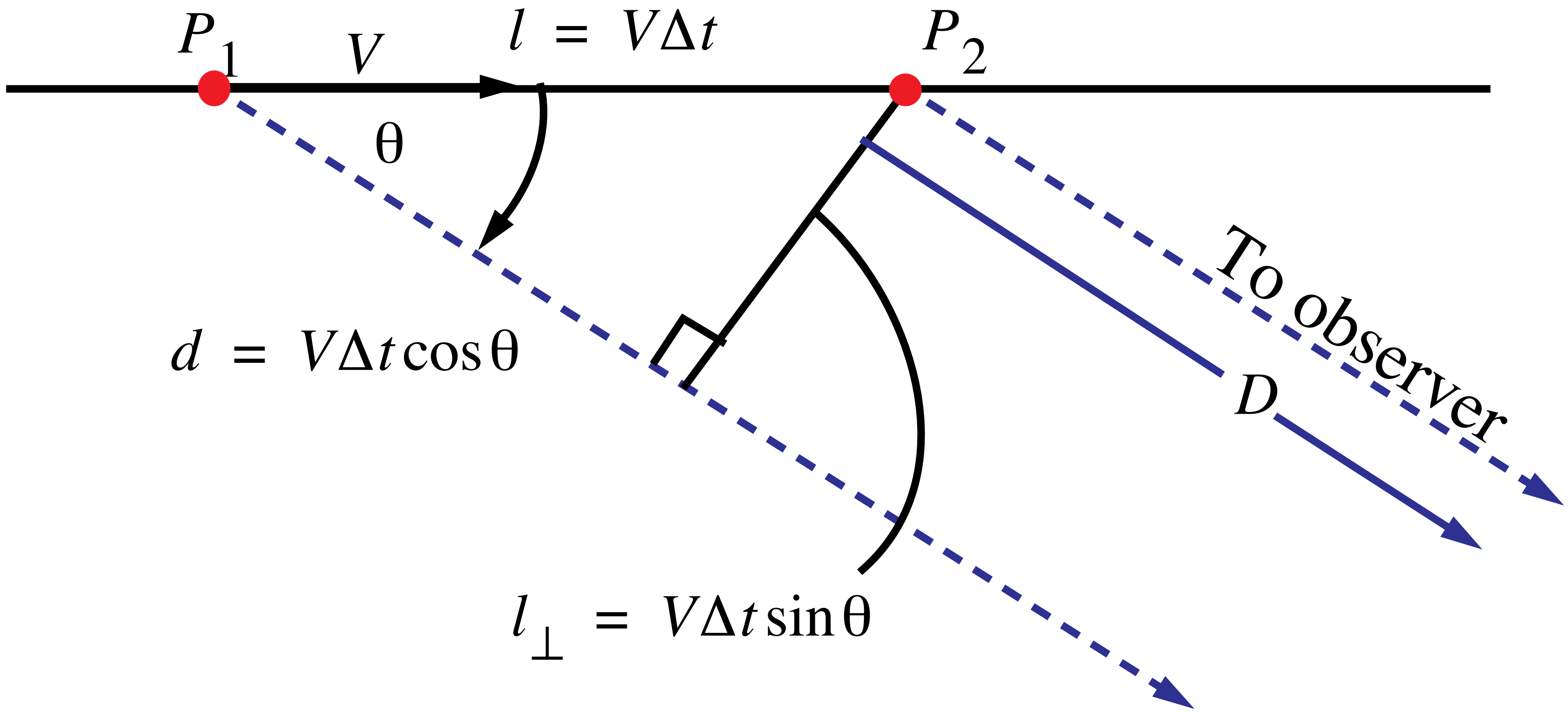
$$\delta = \frac{1}{\Gamma(1-\beta\cos\theta)} \qquad (14)$$

is known as the Doppler factor and figures prominently in the theory of relativistically beamed emission.

2.5 Apparent transverse velocity

Derivation

A relativistic effect which is extremely important in high energy astrophysics and which is analysed in a very similar way to the Doppler effect, relates to the apparent transverse velocity of a relativistically moving object.


P_1
V
$l = V\Delta t$
P_2
θ
To observer
$d = V\Delta t\cos\theta$
D
$l_\perp = V\Delta t\sin\theta$

Consider an object which moves from P_1 to P_2 in a time Δt in the observer's frame. In this case, Δt need not be the time between the beginning and end of a periodic. Indeed, in practice, Δt is usually of order a year. As before, the time difference between the time of receptions of photons emitted at P_1 and P_2 are given by:

$$\Delta t_{\mathrm{rec}} = \Delta t\left(1 - \frac{V}{c}\cos\theta\right) \qquad (15)$$

The apparent distance moved by the object is

$$l_{\perp} = V\Delta t \sin\theta \qquad (16)$$

Hence, the apparent velocity of the object is:

$$V_{\mathrm{app}} = \frac{V \Delta t \sin\theta}{\Delta t\left(1 - \frac{V}{c}\cos\theta\right)} = \frac{V \sin\theta}{\left(1 - \frac{V}{c}\cos\theta\right)}$$

$$\frac{V_{\mathrm{app}}}{c} = \frac{\frac{V}{c}\sin\theta}{\left(1 - \frac{V}{c}\cos\theta\right)} \qquad (17)$$

In terms of

$$\beta_{app} = \frac{V_{app}}{c} \qquad \beta = \frac{V}{c}$$
$$\beta_{app} = \frac{\beta \sin\theta}{1 - \beta\cos\theta} \tag{18}$$

The non-relativistic limit is just $V_{app} = V\sin\theta$, as we would expect. However, note that this result is not a consequence of the Lorentz transformation, but a consequence of light travel time effects as a result of the finite speed of light.

Consequences

For angles close to the line of sight, the effect of this equation can be dramatic. First, determine the angle for which the apparent velocity is a maximum:

$$\frac{d\beta_{\mathrm{app}}}{d\theta} = \frac{(1-\beta\cos\theta)\beta\cos\theta - \beta\sin\theta\beta\sin\theta}{(1-\beta\cos\theta)^2} = \frac{\beta\cos\theta - \beta^2}{(1-\beta\cos\theta)^2} \qquad (19)$$

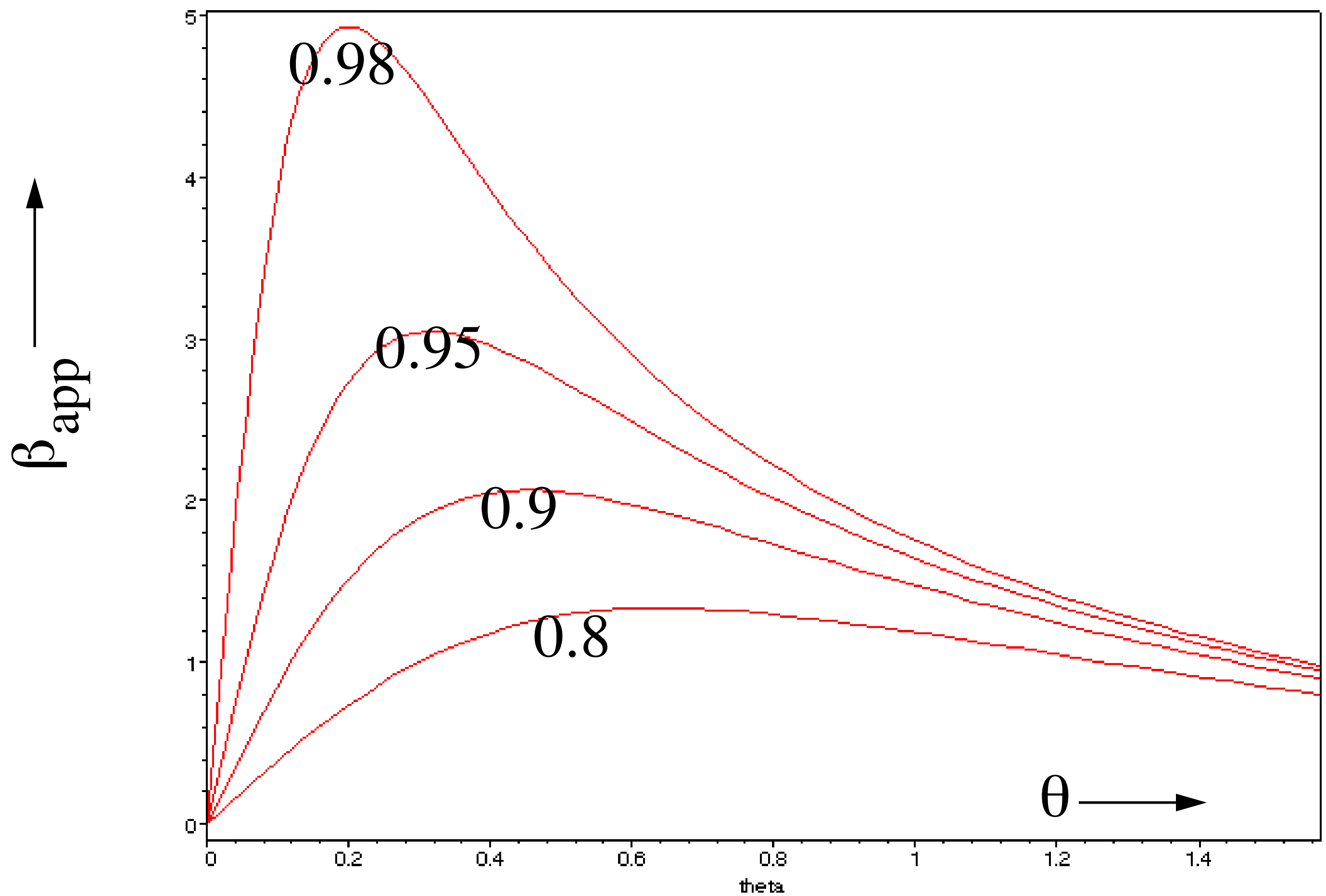
This derivative is zero when

$$\cos\theta = \beta \qquad (20)$$

At the maximum:

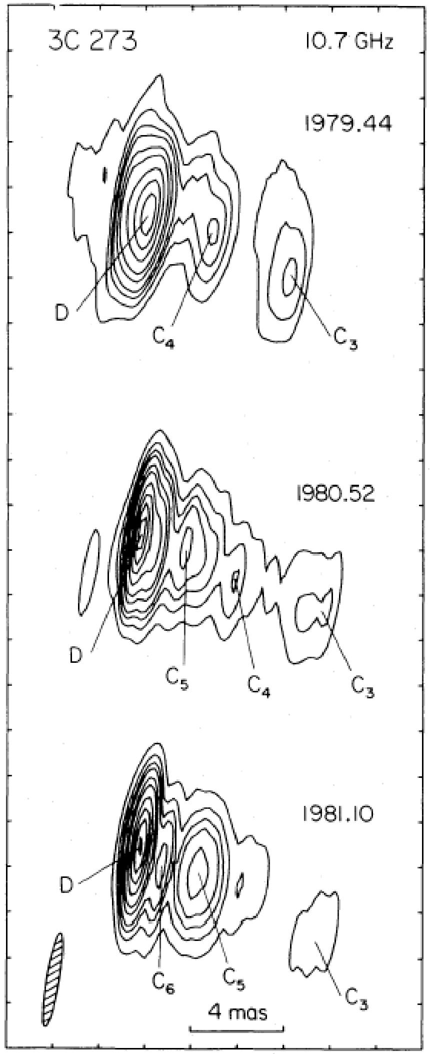
$$\beta_{app} = \frac{\beta\sin\theta}{1-\beta\cos\theta} = \frac{\beta\sqrt{1-\beta^2}}{1-\beta^2} = \frac{\beta}{\sqrt{1-\beta^2}} = \Gamma\beta \quad (21)$$

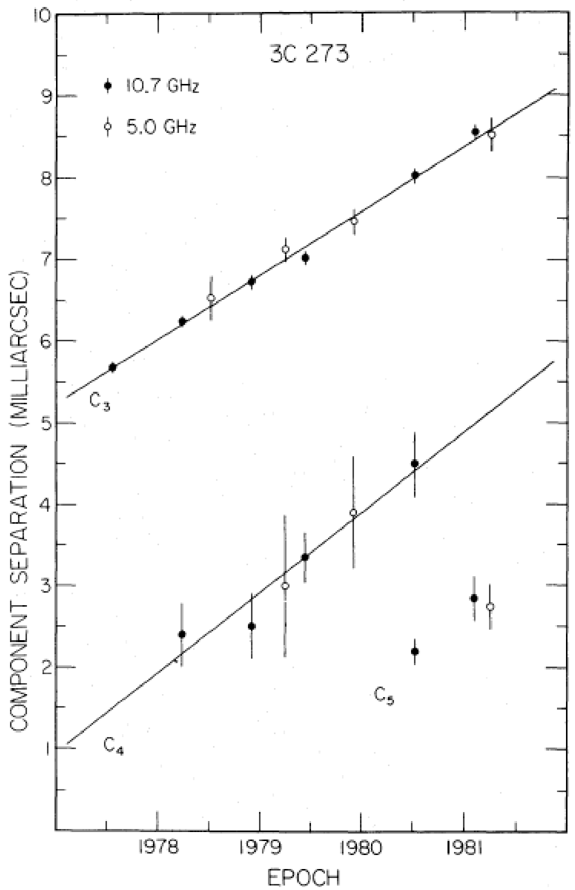
If $\Gamma \gg 1$ then $\beta \approx 1$ and the apparent velocity of an object can be larger than the speed of light. We actually see such effects in AGN. Features in jets apparently move at faster than light speed (after conversion of the angular motion to a linear speed using the redshift of the source.) This was originally used to argue against the cosmological interpretation of quasar redshifts. However, as you can see such large apparent velocities are an easily derived feature of large apparent velocities.



Plots of β_{app} for various indicated values of β as a function of θ.

The following images are from observations of 3C 273 over a period of 5 years from 1977 to 1982. They show proper motions in the knots C_3 and C_4 of 0.79±0.03 mas/yr and 0.99±0.24 mas/yr respectively. These translate to proper motions of $5.5 \pm 0.2 h^{-1} c$ and $6.9 \pm 1.7 h^{-1} c$ respectively.

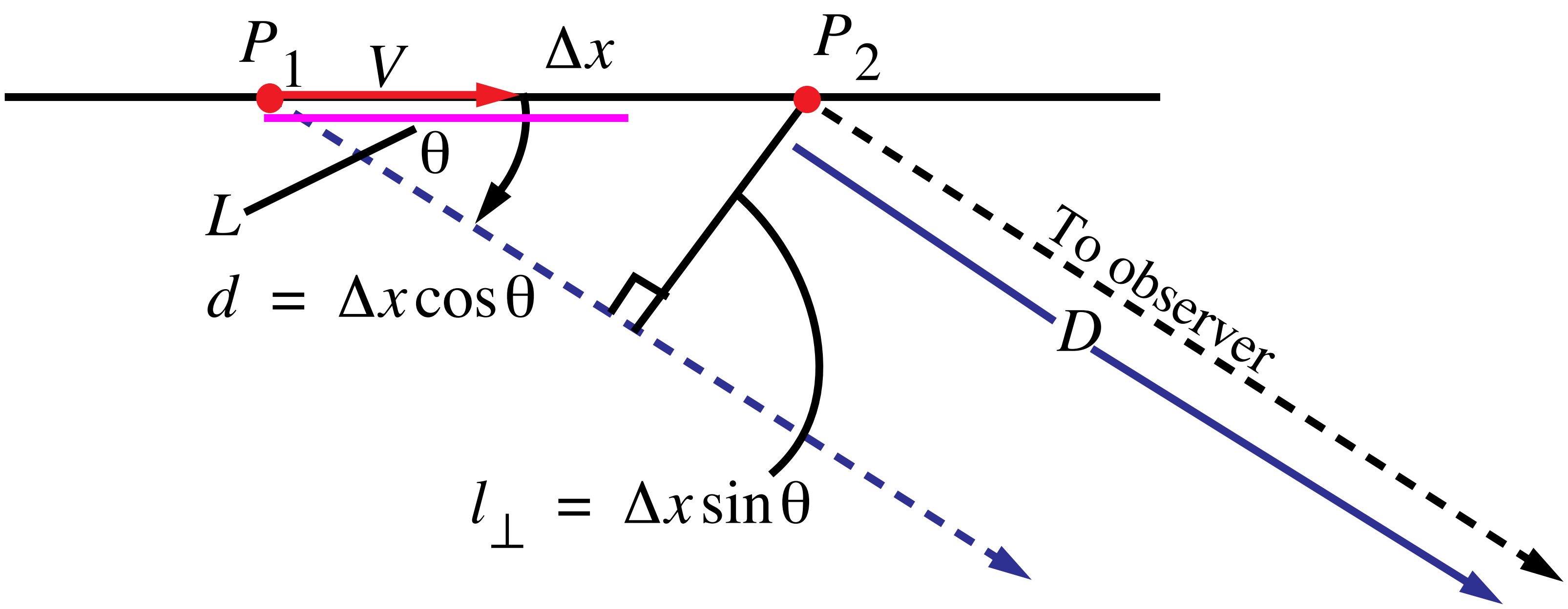




From Unwin et al., ApJ, **289**, 109

2.6 Apparent length of a moving rod

The Lorentz-Fitzgerald contraction gives us the relationship between the proper lengths of moving rods. An additional factor enters when we take into account time retardation.



Consider a rod of length

$$L = \Gamma^{-1} L_0 \tag{22}$$

in the observer's frame. Now the apparent length of the rod is affected by the fact that photons which arrive at the observer at the same time are emitted at different times. P_1 corresponds to when the trailing end of the rod passes at time t_1 and P_2 corresponds to when the leading end of the rod passes at time t_2. Equating the arrival times for photons emitted from P_1 and P_2 at times t_1 and t_2 respectively,

$$
\begin{aligned}
t_1 + \frac{D + \Delta x \cos\theta}{c} &= t_2 + \frac{D}{c} \\
\Rightarrow t_2 - t_1 &= \frac{\Delta x \cos\theta}{c}
\end{aligned}
\qquad (23)
$$

When the trailing end of the rod reaches P_2 the leading end has to go a further distance $\Delta x - L$ which it does in $t_2 - t_1$ secs. Hence,

$$\Delta x - L = \frac{V \Delta x \cos\theta}{c}$$

$$\Rightarrow \Delta x = \frac{L}{1 - \frac{V}{c}\cos\theta} \qquad (24)$$

and the apparent projected length is

$$L_{\mathrm{app}} = \Delta x \sin\theta = \frac{L \sin\theta}{1 - \beta\cos\theta} = \frac{L_0}{\Gamma(1 - \beta\cos\theta)} = \delta L_0 \qquad (25)$$

This is another example of the appearance of the ubiquitous Doppler factor.

2.7 Transformation of velocities

The Lorentz transformation

$$x = \Gamma(x' + Vt') \qquad y = y' \qquad z = z'$$

$$t = \Gamma\left(t' + \frac{Vx'}{c^2}\right) \tag{26}$$

can be expressed in differential form:

$$dx = \Gamma(dx' + Vdt') \qquad dy = dy' \qquad dz = dz'$$

$$dt = \Gamma\left(dt' + \frac{Vdx'}{c^2}\right) \tag{27}$$

so that if a particle moves dx' in time dt' in the frame S' then the corresponding quantities in the frame S are related by the above differentials. This can be used to relate velocities in the 2 frames via

$$\frac{dx}{dt} = \frac{\Gamma(dx' + V dt')}{\Gamma\left(dt' + \frac{V dx'}{c^2}\right)} = \frac{\frac{dx'}{dt} + V}{1 + \frac{V}{c^2}\frac{dx'}{dt}}$$

$$\Rightarrow v_x = \frac{v_x' + V}{1 + \frac{V v_x'}{c^2}} \qquad (28)$$

For the components of velocity transverse to the motion of S',

$$\frac{dy}{dt} = v_y = \frac{dy'}{\Gamma\left(dt' + \frac{V dx'}{c^2}\right)} = \frac{v_y{}'}{\Gamma\left(1 + \frac{V v_x{}'}{c^2}\right)}$$

$$\frac{dz}{dt} = v_z = \frac{dz'}{\Gamma\left(dt' + \frac{V dx'}{c^2}\right)} = \frac{v_z{}'}{\Gamma\left(1 + \frac{V v_x{}'}{c^2}\right)} \qquad (29)$$

In invariant terms (i.e. independent of the coordinate system), take

$$\begin{aligned} v_{\|} &= \text{Component of velocity parallel to } \boldsymbol{V} \\ v_{\perp} &= \text{Component of velocity perpendiciular to } \boldsymbol{V} \end{aligned} \qquad (30)$$

then

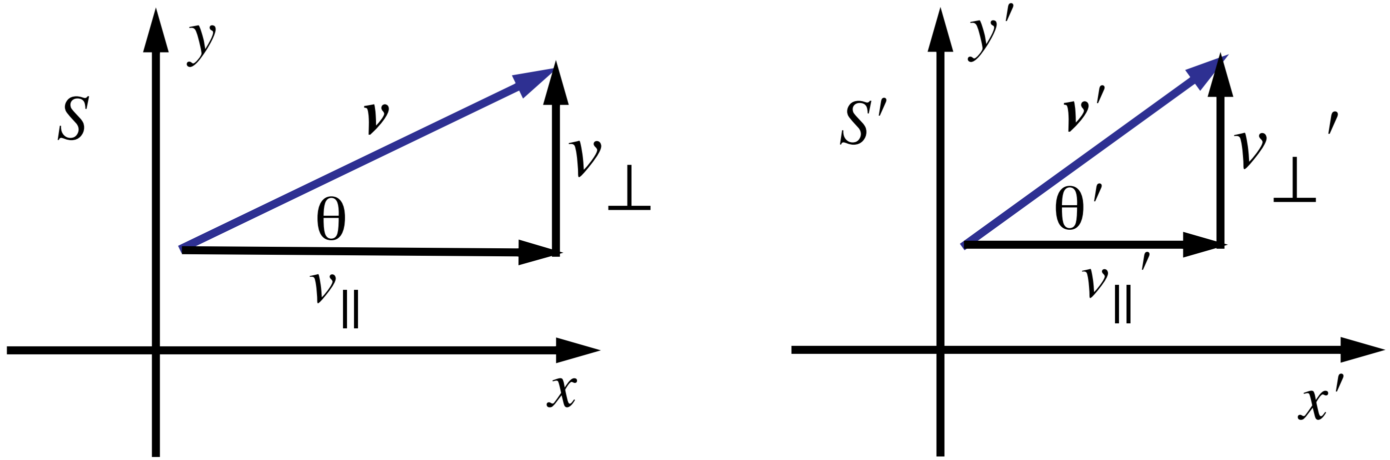
$$v_{\|} = \frac{v_{\|}' + V}{1 + V v_{\|}/c^2} \qquad v_{\perp} = \frac{v_{\perp}'}{\Gamma(1 + V v_{\|}/c^2)} \qquad (31)$$

The reverse transformations are obtained by simply replacing V by $-V$ so that:

$$v_{\|}' = \frac{v_{\|} - V}{1 - V v_{\|}/c^2} \qquad v_{\perp}' = \frac{v_{\perp}}{\Gamma(1 - V v_{\|}/c^2)} \qquad (32)$$

and these can also be recovered by considering the differential form of the reverse Lorentz transformations.

2.8 Aberration



Because of the law of transformation of velocities, a velocity vector make different angles with the direction of motion. From the above laws for transformation of velocities,

$$\tan\theta = \frac{v_{\perp}}{v_{||}} = \frac{v_{\perp}'}{\Gamma(v_{||}' + V)} = \frac{v'\sin\theta}{\Gamma(v'\cos\theta + V)} \qquad (33)$$

(The difference from the non-relativistic case is the factor of Γ.)

The most important case of this is when $v = v' = c$. We put

$$\begin{aligned} v_{||} &= c\cos\theta & v_{\perp} &= c\sin\theta \\ v_{||}' &= c\cos\theta' & v_{\perp}' &= c\sin\theta' \end{aligned} \qquad (34)$$

and

$$\beta = \frac{V}{c} \tag{35}$$

and the angles made by the light rays in the two frames satisfy:

$$c\cos\theta = \frac{c\cos\theta' + V}{1 + \frac{V}{c}\cos\theta'} \Rightarrow \cos\theta = \frac{\cos\theta' + \beta}{1 + \beta\cos\theta'}$$

$$c\sin\theta = \frac{c\sin\theta'}{\Gamma\left(1 + \frac{V}{c}\cos\theta'\right)} \Rightarrow \sin\theta = \frac{\sin\theta'}{\Gamma(1 + \beta\cos\theta')} \tag{36}$$

Half-angle formula

There is a useful expression for aberration involving half-angles. Using the identity,

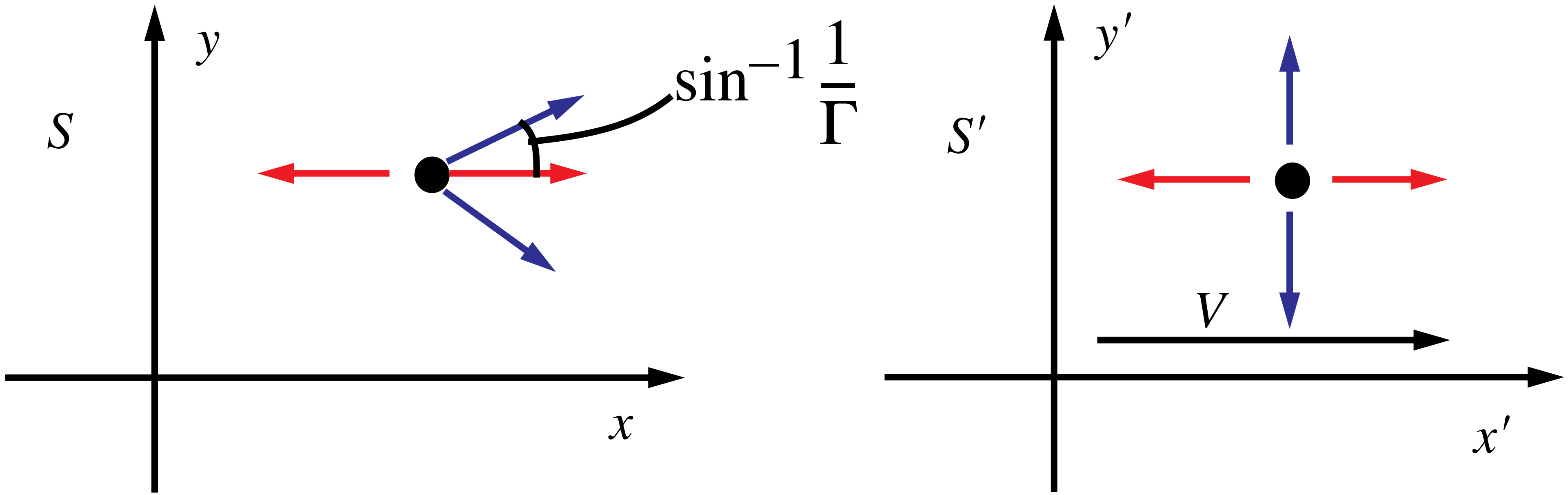
$$\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta} \tag{37}$$

the aberration formulae can be written as:

$$\tan\left(\frac{\theta}{2}\right) = \left(\frac{1-\beta}{1+\beta}\right)^{1/2}\tan\left(\frac{\theta'}{2}\right) \tag{38}$$

Isotropic radiation source

Consider a source of radiation which emits isotropically in its rest frame and which is moving with velocity V with respect to an observer (in frame S). The source is at rest in S' which is moving with velocity V with respect to S.



Consider a rays emitted at right angles to the direction of motion. This has $\theta = \pm\frac{\pi}{2}$. The angle of these rays in S are given by

$$\sin\theta = \pm\frac{1}{\Gamma} \quad (39)$$

$$(40)$$

These rays enclose half the light emitted by the source, so that in the reference frame of the observer, half of the light is emitted in a forward cone of half-angle $1/\Gamma$. This is relativistic beaming in another form. When Γ is large:$\theta \approx \frac{1}{\Gamma}$.

3 Four vectors

3.1 Four dimensional space-time

Special relativity defines a four dimensional space-time continuum with coordinates

$$x^0 = ct \qquad x^1 = x \qquad x^2 = y \qquad x^3 = z \qquad (41)$$

An event is the point in space-time with coordinates x^μ where $\mu = 0, 1, 2, 3$.

The summation convention

Wherever there are repeated upper and lower indices, summation is implied, e.g.

$$A_\mu B^\mu = \sum_{\mu=0}^{3} A_\mu B^\mu \qquad (42)$$

The metric of space-time is given by

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \qquad (43)$$

where

$$\eta_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{Inverse} = \eta^{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (44)$$

(45)

Hence the metric

$$ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \qquad (46)$$

This metric is unusual for a geometry in that it is not positive definite. For *spacelike* displacements it is positive and for *timelike* displacements it is negative.

This metric is related to the proper time τ by

$$ds^2 = -c^2 d\tau^2 \qquad (47)$$

Indices are raised and lowered with $\eta_{\mu\nu}$, e.g. if A^μ is a vector, then

$$A_{\mu} = \eta_{\mu\nu} A^{\nu} \tag{48}$$

This extends to tensors in space-time etc. Upper indices are referred to as *covariant*; lower indices as contravariant.

3.2 Representation of a Lorentz transformation

A Lorentz transformation is a transformation which preserves ds^2. We represent a Lorentz transformation by:

$$\begin{aligned} x^{\mu\prime} &= \Lambda^{\mu}{}_{\nu} x^{\nu} \\ \eta_{\mu\nu} &= \Lambda^{\sigma}{}_{\mu} \Lambda^{\tau}{}_{\nu} \eta_{\sigma\tau} \end{aligned} \tag{49}$$

That is, a Lorentz transformation is the equivalent of an orthogonal matrix in the 4-dimensional space time with indefinite metric.

Conditions:

- $\det\Lambda = 1$ – rules out reflections ($x \to -x$)
- $\Lambda^0{}_0 > 0$ – isochronous

For the special case of a Lorentz transformation involving a *boost* along the x–axis

$$\Lambda^{\mu}{}_{\nu} = \begin{bmatrix} \gamma & -\beta\Gamma & 0 & 0 \\ -\beta\Gamma & \Gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (50)$$

3.3 Some important 4-vectors

The 4-velocity

This is defined by

$$u^{\mu} = \frac{dx^{\mu}}{d\tau} = \left[\frac{dx^0}{d\tau}, \frac{dx^i}{d\tau}\right] \qquad (51)$$

The zeroth component

$$\frac{dx^0}{d\tau} = c\frac{dt}{d\tau} = c\frac{dt}{\sqrt{dt^2 + -c^{-2}[(dx^1)^2 + (dx^2)^2 + (dx^3)^2]}}$$

$$= \frac{c}{\sqrt{1 - \frac{v^2}{c^2}}} = c\gamma \qquad (52)$$

Note that we use Γ for the Lorentz factor of the transformation and γ for particles. This will later translate into Γ for bulk motion and γ for the Lorentz factors of particles in the rest-frame of the plasma.

The spatial components:

$$\frac{dx^i}{d\tau} = \frac{dx^i}{dt}\frac{dt}{d\tau} = \gamma v^i \qquad (53)$$

so that

$$u^\mu = [\gamma c, \gamma v^i] \qquad (54)$$

The 4-momentum

The 4-momentum is defined by

$$p^\mu = m_0 u^\mu = [\gamma mc, \gamma mv^i] = \left[\frac{E}{c}, p^i\right] \qquad (55)$$

where

$$E = \sqrt{c^2 p^2 + m^2 c^4} = \gamma m c^2 \qquad (56)$$

is the energy, and

$$p^i = \gamma m_0 v^i \qquad (57)$$

is the 3-momentum.

Note the magnitude of the 4-momentum

$$\begin{aligned} \eta_{\mu\nu} p^\mu p^\nu &= -(p^0)^2 + (p^1)^2 + (p^2)^2 + (p^3)^2 \\ &= -\left(\frac{E}{c}\right)^2 + p^2 = -m^2 c^2 \end{aligned} \qquad (58)$$

3.4 Transformation of 4-vectors

Knowing that a quantity is a 4-vector means that we can easily determine its behaviour under the effect of a Lorentz transformation. The zero component behaves like x^0 and the x component behaves like x. Recall that:

$$x = \Gamma(x' + \beta x^{0\prime}) \qquad x^0 = \Gamma(x^{0\prime} + \beta x') \tag{59}$$

Therefore, the components of the 4-velocity transform like

$$\begin{aligned} U^{0\prime} &= \Gamma(U^0 - \beta U^1) \\ U^{1\prime} &= \Gamma(-\beta U^0 + U^1) \end{aligned} \tag{60}$$

Hence,

$$c\gamma' = \Gamma(c\gamma - \beta\gamma v^1) \Rightarrow \gamma' = \Gamma\gamma\left(1 - \beta\frac{v^1}{c}\right)$$

$$\gamma' v^{1\prime} = \Gamma(-\beta c\gamma + \gamma v^1) \Rightarrow \gamma' v^{1\prime} = \Gamma\gamma(v^1 - c\beta) \qquad (61)$$

$$\gamma' v^{2\prime} = \gamma v^2$$

$$\gamma' v^{3\prime} = \gamma v^3$$

Transformation of Lorentz factors

Putting $v^1 = v\cos\theta$ gives

$$\gamma' = \Gamma\gamma\left(1 - \beta\frac{v}{c}\cos\theta\right) \qquad (62)$$

This is a useful relationship that can be derived from the previous transformations for the 3-velocity. However, one of the useful features of 4-vectors is that this transformation of the Lorentz factor is easily derived with little algebra.

Transformation of 3-velocities

Dividing the second of the above transformations by the first:

$$v^{1\prime} = \frac{\Gamma\gamma(v^1 - c\beta)}{\Gamma\gamma\left(1 - \beta\frac{v^1}{c}\right)} = \frac{(v^1 - c\beta)}{\left(1 - \beta\frac{v^1}{c}\right)} = \frac{(v^1 - V)}{\left(1 - \frac{Vv_x}{c^2}\right)} \qquad (63)$$

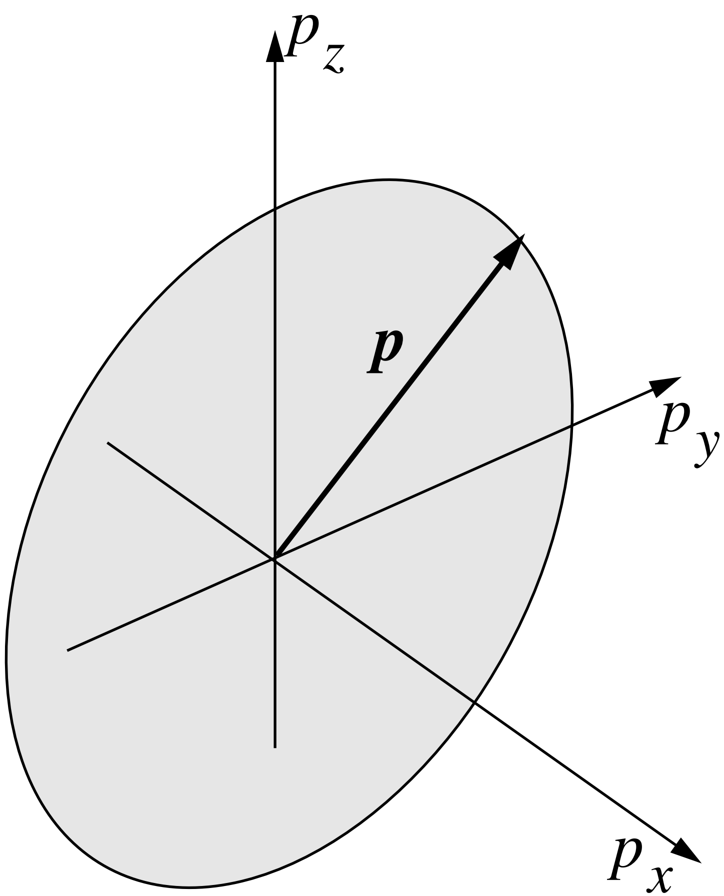
Dividing the third equation by the first:

$$v^{2\prime} = \frac{\gamma v^2}{\Gamma\gamma\left(1 - \beta\frac{v^1}{c}\right)} = \frac{v^2}{\Gamma\left(1 - \frac{Vv_x}{c^2}\right)} \qquad (64)$$

and similarly for v^3. These are the equations for the transformation of velocity components derived earlier.

4 Distribution functions in special relativity

In order to properly describe distributions of particles in a rel-



Distribution of momenta in momentum space.

ativistic context and in order to understand the transformations of quantities such as specific intensity, etc. we need to have relativistically covariant descriptions of statistical distributions of particles.

Recall the standard definition of the phase space distribution function:

$$f d^3x d^3p = \begin{array}{c}\text{No of particles within an elementrary volume} \\ \text{of phase space}\end{array} \tag{65}$$

4.1 Momentum space and invariant 3-volume

The above definition of $f(\boldsymbol{x}, t, \boldsymbol{p})$ is somewhat unsatisfactory from a relativistic point of view since it focuses on three dimensions rather than four.

Covariant analogue of d^3p

The aim of the following is to replace d^3p by something that makes sense relativistically.

Consider the space of 4-dimensional momenta. We express the components of the momentum in terms of a hyperspherical angle χ and polar angles θ and ϕ.

$$
\begin{aligned}
p^0 &= mc\cosh\chi \\
p^1 &= mc\sinh\chi\sin\theta\cos\phi \\
p^2 &= mc\sinh\chi\sin\theta\sin\phi \\
p^3 &= mc\sinh\chi\cos\theta
\end{aligned}
\qquad (66)
$$

The Minkowksi metric is also the metric of momentum space and we express the interval between neighbouring momenta as $\eta_{\mu\nu}dp^{\mu}dp^{\nu}$. In terms of hyperspherical angles:

$$
\begin{aligned}
\eta_{\mu\nu}dp^{\mu}dp^{\nu} &= -(d(mc))^2 + \\
&\quad + m^2c^2[(d\chi)^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta d\phi^2)]
\end{aligned}
\qquad (67)
$$

This is proved in Appendix A.

The magnitude of the 3-dimensional momentum is

$$p = mc\sinh\chi \qquad (68)$$

A particle of mass m is restricted to the *mass shell* $m = \text{constant}$. This is a 3-dimensional hypersurface in momentum space. From the above expression for the metric, it is easy to read off the element of volume on the mass shell:

$$d\omega = (mc)^3 \sinh^2\chi \sin\theta d\theta d\phi \qquad (69)$$

This volume is an ***invariant*** since it corresponds to the invariantly defined subspace of the momentum space, $m = \text{constant}$.

On the other hand, the volume element d^3p refers to a sub-space which is not invariant. The quantity

$$d^3p = dp^1dp^2dp^3 \qquad (70)$$

depends upon the particular Lorentz frame. It is in fact the projection of the mass shell onto $p^0 =$ constant. However, it is useful to know how the expression for d^3p is expressed in terms of hyperspherical coordinates.

In the normal polar coordinates:

$$d^3p = p^2 \sin\theta dp d\theta d\phi \qquad (71)$$

Putting $p = mc\sinh\chi$ in this expression,

$$d^3 p = (mc)^3 \sinh^2\chi \cosh\chi \sin\theta \, d\chi \, d\theta \, d\phi = \cosh\chi \, d\omega \quad (72)$$

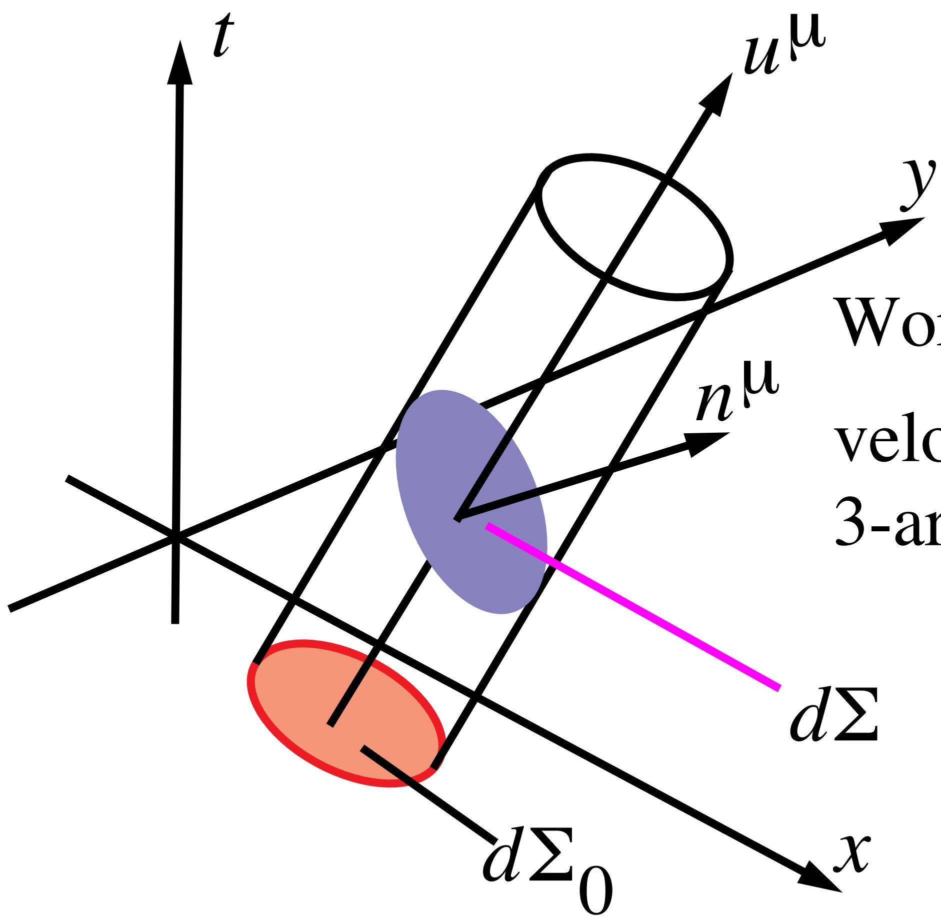
That is, the normal momentum space 3-volume and the invariant volume $d\omega$ differ by a factor of $\cosh\chi$.

4.2 Invariant definition of the distribution function

The following invariant expression of the distribution function was first introduced by J.L. Synge who was one of the influential pioneers in the theory of relativity who introduced geometrical and invariant techniques to the field.

We begin by defining a world tube of particles with momentum p^μ (4-velocity u^μ).

The distribution function $f(x^\mu, p^\mu)$ is defined by:



World tube of particles with 4-velocity u^μ. The cross-sectional 3-area of the tube is $d\Sigma_0$.

The three-area of the world tube, $d\Sigma_0$ is the particular 3-area that is normal to the world lines in the tube. Using $d\Sigma_0$, we define the distribution function, $f(x^\mu, p^\mu)$, by the following definition:

$$\begin{array}{c}\text{Number of world lines within}\\ \text{the world tube with momenta}\\ \text{within } d\omega\end{array} = f(x^\mu, p^\mu)\, d\Sigma_0\, d\omega \qquad (73)$$

This is expressed in terms of a particular 3-area, $d\Sigma_0$.

Now consider the world lines intersecting an arbitrary 3-area (or 3-volume) $d\Sigma$ that has a unit normal n^{μ}. The projection relation between $d\Sigma$ and $d\Sigma_0$ is

$$d\Sigma_0 = d\Sigma \times (-c^{-1} u^{\mu} n_{\mu}) \qquad (74)$$

Proof of last statement

First, let us define what is meant by a spacelike hypersurface. In such a hypersurface every displacement, dx^{μ}, is spacelike. That is, $\eta_{\mu\nu} dx^{\mu} dx^{\nu} > 0$. The normal to a spacelike hypersurface is timelike. The square of the magnitude of a unit normal is -1:

$$\eta_{\mu\nu} n^{\mu} n^{\nu} = -1 \tag{75}$$

Example:

The surface

$$t = \text{constant} \tag{76}$$

is spacelike. Its unit normal is:

$$n^{\mu} = (1, 0, 0, 0) \tag{77}$$

We can also contemplate a family of spacelike hypersurfaces in which, for example

$$t = \text{variable} \tag{78}$$

This corresponds to a set of 3-volumes in which x^1, x^2 and x^3 vary, that are swept along in the direction of the time-axis. As before the unit normal to this family of hypersurfaces is $n^\mu = (1, 0, 0, 0)$ and the corresponding 4-velocity is

$$u^\mu = \left(c\frac{dt}{d\tau}, 0, 0, 0\right) = (c, 0, 0, 0) = cn^\mu \qquad (79)$$

In the present context, we can consider the set of 3-spaces $d\Sigma_0$ corresponding to each cross-section of a world tube as a family of such spacelike hypersurfaces. Each hypersurface is defined as being perpendicular to the 4-velocity so that, in general, $u_0^\mu \neq (c, 0, 0, 0)$. However, it is possible to make a Lorentz transformation so that in a new system of coordinates $n_0^\mu = (1, 0, 0, 0)$ and $u_0^\mu = (c, 0, 0, 0)$.

The significance of $d\Sigma$

What is the significance of a surface $d\Sigma$ as indicated in the figure? This is an arbitrary surface tilted with respect to the original cross-sectional surface $d\Sigma_0$.This surface has its own unit normal n_Σ^μ and 4-velocity, $u_\Sigma^\mu = c n_\Sigma^\mu$.

In the coordinate system in which the normal to $d\Sigma_0$ has components $n_0^\mu = (1, 0, 0, 0)$, let us assume that the 4-velocity of $d\Sigma$ is $u_\Sigma^\mu = (\gamma c, \gamma \boldsymbol{v})$. That is, $d\Sigma$ represents a surface that is moving with respect to $d\Sigma_0$ at the velocity $\boldsymbol{v}$ with Lorentz factor, γ. The unit normal to $d\Sigma$ is

$$n_\Sigma^\mu = (\gamma, \gamma\boldsymbol{\beta}) \tag{80}$$

Relation between volumes in the two frames

Let $d\Sigma$ be the primed (moving) frame. The element of volume of $d\Sigma$ is

$$d\Sigma = dx^{1'} dx^{2'} dx^{3'} \tag{81}$$

At an instant of time in the primed frame denoted by $dt' = 0$

$$\begin{aligned} dx^1 &= \gamma(dx^{1'} + v dt') = \gamma dx^{1'} \\ dx^2 &= dx^{2'} \qquad dx^3 = dx^{3'} \end{aligned} \tag{82}$$

Hence,

$$d\Sigma_0 = dx^1 dx^2 dx^3 = \gamma dx^{1'} dx^{2'} dx^{3'} = \gamma d\Sigma \tag{83}$$

Expression of the Lorentz factor in invariant form

Consider the invariant scalar product

$$n_{\Sigma}^{\mu} n_{0,\,\mu} = (\gamma, \gamma\beta) \cdot (-1, 0, 0, 0) = -\gamma \qquad (84)$$

Dropping the Σ subscript on n_{Σ}^{μ} and using $n_{0,\,\mu} = c^{-1} u^{\mu}$ we have

$$\gamma = -c^{-1} u^{\mu} n_{\mu} \qquad (85)$$

Hence,

$$d\Sigma_0 = (-c^{-1} u^{\mu} n_{\mu}) d\Sigma \qquad (86)$$

Note that the "projection factor" is greater than unity, perhaps counter to intuition.

Number of world lines in terms of $d\Sigma$

We defined the distribution function by:

$$\begin{array}{c}\text{Number of world lines within} \\ \text{the world tube with momenta} \\ \text{within } d\omega\end{array} = f(x^\mu, p^\mu) d\Sigma_0 d\omega \quad (87)$$

Hence, our new definition for an arbitrary $d\Sigma$:

$$\begin{array}{c}\text{Number of world lines within} \\ \text{the world tube crossing } d\Sigma \\ \text{with momenta within } d\omega\end{array} = f(x^\mu, p^\mu)(-c^{-1} u^\mu n_\mu) d\Sigma d\omega$$

Counting is an invariant operation and all of the quantities appearing in the definition of f are invariants, therefore

$$f(x^{\mu}, p^{\mu}) = \text{Invariant} \qquad (88)$$

4.3 An important special case

Take the normal to Σ to be parallel to the time direction in an arbitrary Lorentz frame. Then

$$n^{0} = (1, 0, 0, 0) \qquad (89)$$

and

$$-u_{\mu}n^{\mu} = -(-u^{0}n^{0}) = u^{0} \qquad (90)$$

Also

$$d\Sigma = d^3x \tag{91}$$

Now

$$p^0 = mc\cosh\chi \Rightarrow u^0 = \cosh\chi \tag{92}$$

Therefore,

$$\begin{aligned} f(x^\mu, p^\mu)(-u^\mu n_\mu)d\Sigma d\omega &= f(x^\mu, p^\mu)\cosh\chi d^3x d\omega \\ &= f(x^\mu, p^\mu)d^3x d^3p \end{aligned} \tag{93}$$

Our invariant expression reduces to the noninvariant expression when we select a special 3-volume in spacetime. Thus the usual definition of the distribution is Lorentz-invariant even though it does not appear to be.

5 Distribution of photons

5.1 Definition of distribution function

We can treat massless particles separately or as a special case of the above, where we let $m \to 0$ and $\cosh\chi \to \infty$ in such a way that $mc\cosh\chi \to \frac{h\nu}{c}$. In either case, we have for photons,

$$f(x^\mu, p^\mu)d^3x d^3p = \begin{array}{c}\text{No of photons within } d^3x \\ \text{and momenta within } d^3p\end{array} \qquad (94)$$

and the distribution function is still an invariant.

5.2 Relation to specific intensity

From the definition of the distribution function, we have

$$\begin{array}{c}\text{Energy density of photons} \\ \text{within } d^3p\end{array} = h\nu f d^3p = h\nu f p^2 dp d\Omega \qquad (95)$$

The alternative expression for this involves the energy density per unit frequency per unit solid angle, $u_\nu(\Omega)$. We know that

$$u_\nu(\Omega) = \frac{I_\nu}{c} \qquad (96)$$

Hence, the energy density within ν and within solid angle $d\Omega$ is

$$u_\nu d\nu d\Omega = c^{-1} I_\nu d\nu d\Omega \qquad (97)$$

Therefore,

$$h\nu f p^2 dp d\Omega = h\nu f\left(\frac{h\nu}{c}\right)^2 d\left(\frac{h\nu}{c}\right) d\Omega = c^{-1} I_\nu d\nu d\Omega$$

$$\Rightarrow f = \frac{c^2}{h^4}\frac{I_\nu}{\nu^3} \qquad (98)$$

This gives the very important result that, since f is a Lorentz invariant, then

$$\frac{I_\nu}{\nu^3} = \text{Lorentz invariant} \qquad (99)$$

Thus, if we have 2 relatively moving frames, then

$$\frac{I_{\nu'}}{(\nu')^3} = \frac{I_\nu}{\nu^3} \Rightarrow I_\nu = \left(\frac{\nu}{\nu'}\right)^3 I_{\nu'} \tag{100}$$

Take the primed frame to be the rest frame, then

$$I_\nu = \delta^3 I_{\nu'} \tag{101}$$

where δ is the Doppler factor.

5.3 Transformation of emission and absorption coefficients

Consider the radiative transfer equation:

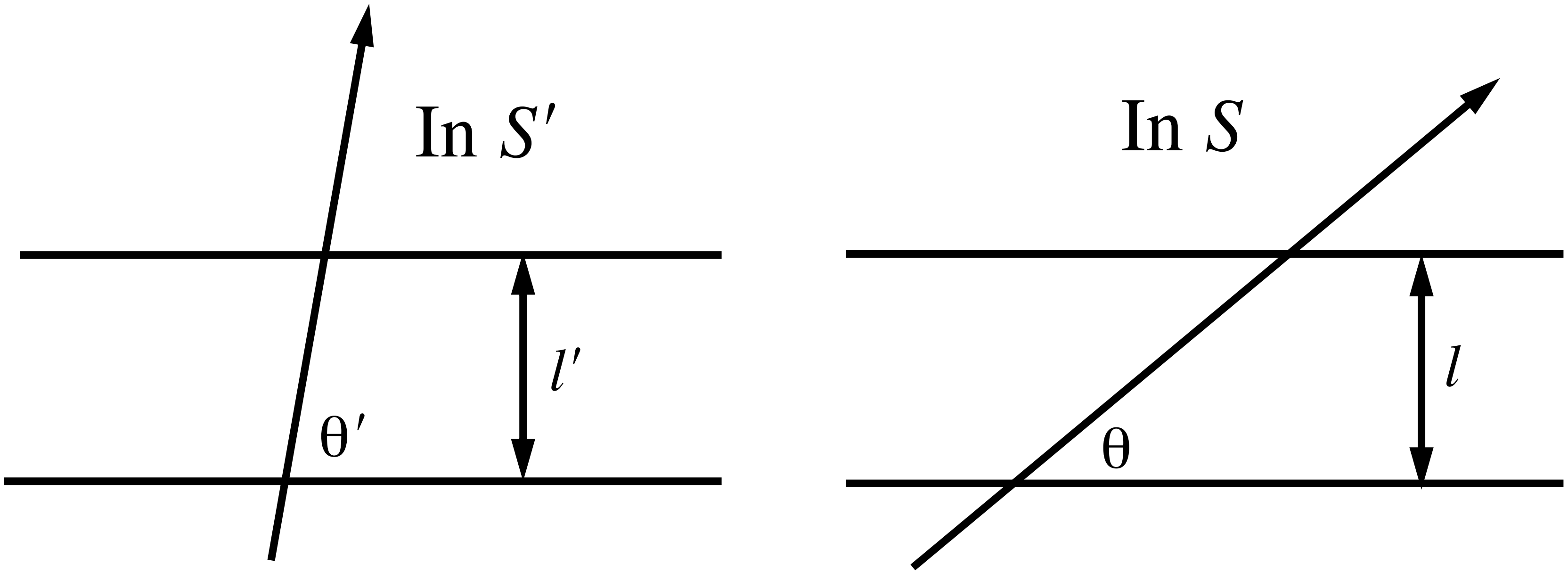
$$\frac{dI_\nu}{d\tau_\nu} = S_\nu - I_\nu \qquad S_\nu = \frac{j_\nu}{\alpha_\nu} \tag{102}$$

Obviously, the source function must have the same transformation properties as I_ν. Hence

$$\frac{S_\nu}{\nu^3} = \text{Lorentz invariant} \tag{103}$$

Emission coefficient

The optical depth along a ray passing through a medium with absorption coefficient α_ν is, in the primed frame



$$\tau = \frac{l'\alpha_{\nu'}}{\sin\theta'} \qquad (104)$$

The optical depth in the unprimed frame is

$$\tau = \frac{l\alpha_\nu}{\sin\theta} \tag{105}$$

and is identical. The factor $e^{-\tau}$ counts the number of photons absorbed so that τ is a Lorentz invariant. Hence

$$\frac{l\alpha_\nu}{l'\alpha_{\nu'}}\frac{\sin\theta'}{\sin\theta} = 1 \tag{106}$$

The aberration formula gives

$$\sin\theta' = \frac{\sin\theta}{\Gamma(1-\beta\cos\theta)} = \delta\sin\theta \tag{107}$$

and the lengths l and l' are perpendicular to the motion, so that $l = l'$. Hence,

$$\frac{\alpha_\nu}{\alpha_{\nu'}} = \delta^{-1} \rightarrow \frac{\nu\alpha_\nu}{\nu'\alpha_{\nu'}} = 1 \qquad (108)$$

i.e. $\nu\alpha_\nu$ is a Lorentz invariant.

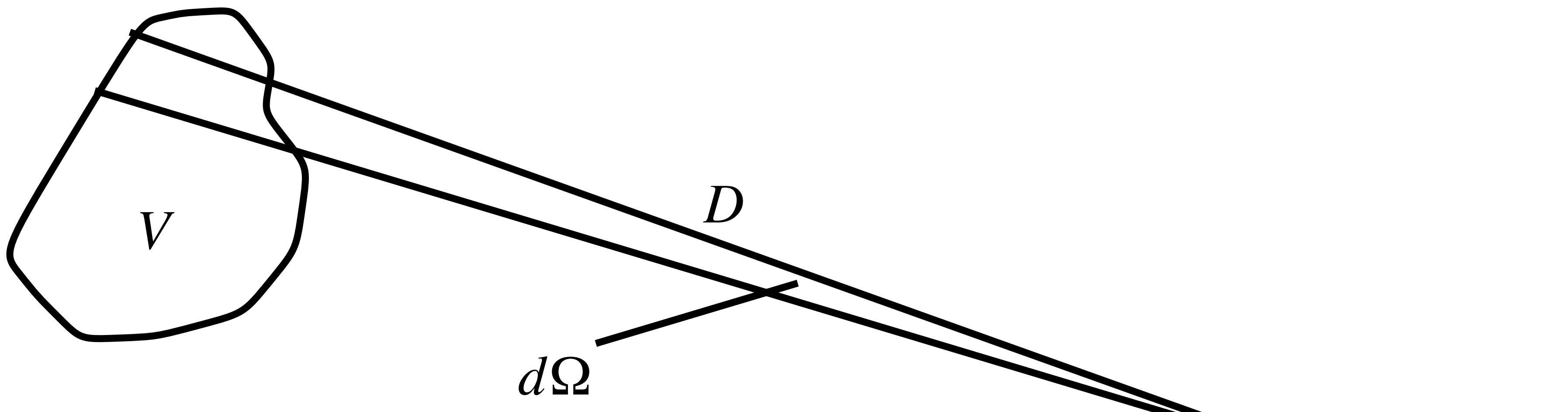
The emission coefficient

$$\frac{S_\nu}{\nu^3} = \frac{j_\nu}{\nu^3\alpha_\nu} = \left(\frac{j_\nu}{\nu^2}\right)(\nu\alpha_\nu)^{-1} = \text{Lorentz invariant} \qquad (109)$$

Hence, j_ν / ν^2 is a Lorentz invariant.

5.4 Flux density from a moving source

The flux from an arbitrary source is given by



$$F_\nu = \int_\Omega I_\nu \cos\theta \, d\Omega \approx \int_\Omega I_\nu \, d\Omega = \frac{1}{D^2} \int_V j_\nu \, dV \qquad (110)$$

Now relate this to the emissivity in the rest frame. Since

$$\frac{j_\nu}{\nu^2} = \text{Lorentz invariant}$$
$$j_\nu = \left(\frac{\nu}{\nu'}\right)^2 j'_{\nu'} = \delta^2 j'_{\nu'} \tag{111}$$

Therefore,

$$F_\nu = \frac{1}{D^2}\int_V \delta^2 j'_{\nu'} dV = \frac{\delta^2}{D^2}\int_V j'_{\nu'} dV \tag{112}$$

The apparent volume of the source is related to the volume in the rest frame, by

$$dV = \delta dV' \qquad (113)$$

This is the result of a factor of δ expansion in the direction of motion and no expansion in the directions perpendicular to the motion. Hence the flux density is given in terms of the rest frame parameters by:

$$F_\nu = \frac{1}{D^2}\int_V \delta^2 j'_{\nu'} dV = \frac{\delta^3}{D^2}\int_V j'_{\nu'} dV' \qquad (114)$$

Effect of spectral index

For a power-law emissivity (e.g. synchrotron radiation),

$$j_{\nu'} = j'_{\nu}\left(\frac{\nu'}{\nu}\right)^{-\alpha} = \delta^{\alpha} j'_{\nu} \tag{115}$$

Therefore,

$$F_{\nu} = \frac{\delta^{3+\alpha}}{D^2} \int_V j'_{\nu} dV' \tag{116}$$

This gives a factor of $\delta^{3+\alpha}$ increase for a blue-shifted source of radiation, over and above what would be measured in the rest frame at the same frequency.

Example:

Consider the beaming factor in a $\Gamma = 5$ jet viewed at the angle which maximises the apparent proper motion.

The maximum β_{app} occurs when $\cos\theta = \beta$.

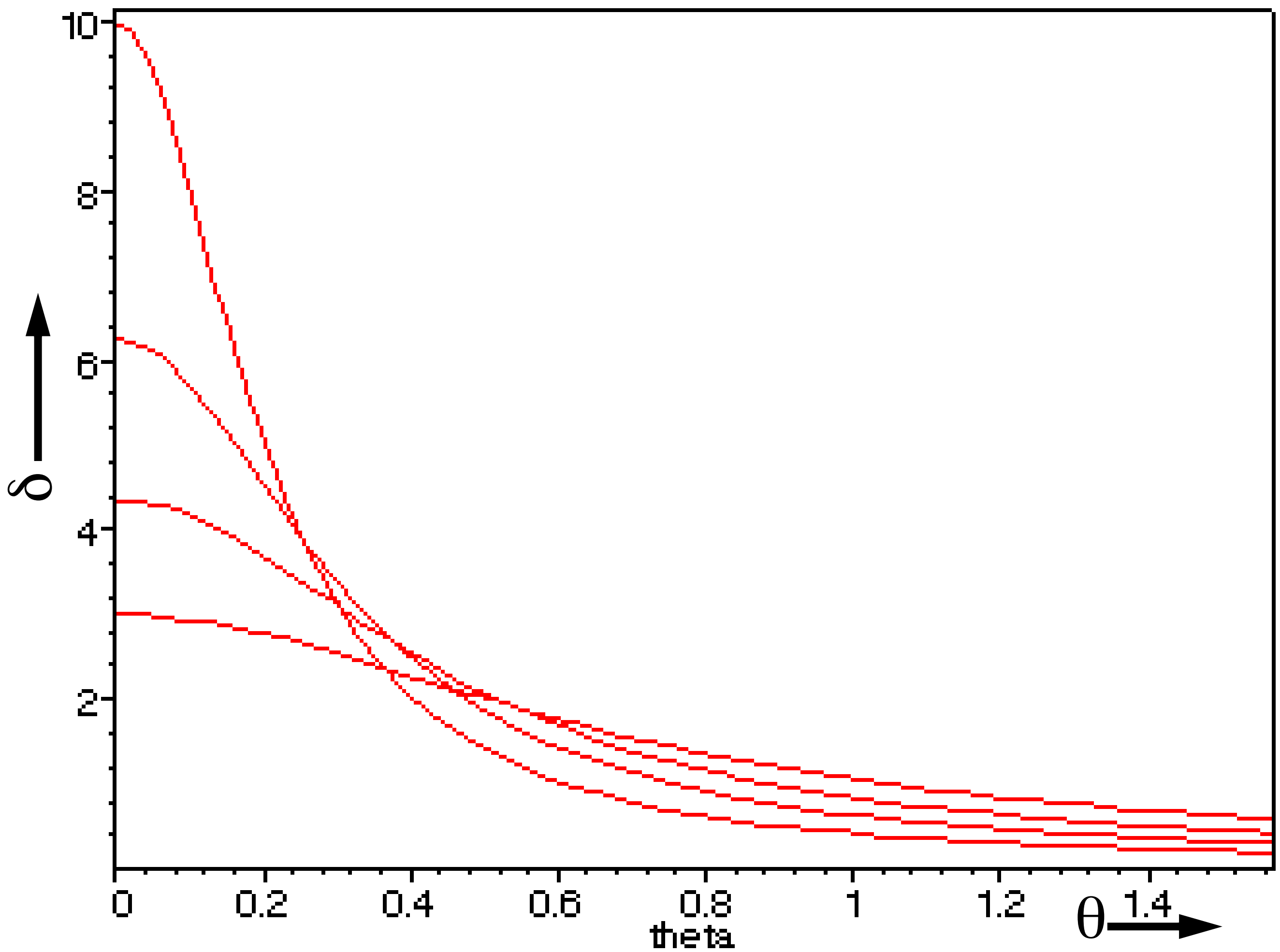
$$\Gamma = 5 \Rightarrow \beta = \sqrt{1 - \frac{1}{5^2}} = 0.9798$$
$$\delta = \frac{1}{\Gamma(1 - \beta\cos\theta)} = \frac{1}{\Gamma(1 - \beta^2)} = \Gamma \qquad (117)$$

Hence,

$$\delta^{3+\alpha} = 5^{3.6} = 330 \qquad (118)$$

for a spectral index of 0.6

5.5 Plot of δ



Plot of the Doppler factor as a function of viewing angle.

Appendix A

Line element in momentum space in terms of hyper-spherical angles

We have the hyperspherical angle representation of a point in momentum space:

$$\begin{aligned} p^0 &= mc\cosh\chi \\ p^1 &= mc\sinh\chi\sin\theta\cos\phi \\ p^2 &= mc\sinh\chi\sin\theta\sin\phi \\ p^3 &= mc\sinh\chi\cos\theta \end{aligned} \qquad (119)$$

We can write:

$$\begin{bmatrix} dp^0 \\ dp^1 \\ dp^2 \\ dp^3 \end{bmatrix} = A \begin{bmatrix} d(mc) \\ d\chi \\ d\theta \\ d\phi \end{bmatrix} \qquad (120)$$

where

$$A = \begin{bmatrix} \cosh\chi & mc\sinh\chi & 0 & 0 \\ \sinh\chi & mc\cosh\chi & mc\sinh\chi & -mc\sinh\chi \\ \times\sin\theta\cos\phi & \times\sin\theta\cos\phi & \times\cos\theta\cos\phi & \times\sin\theta\cos\phi \\ \sinh\chi & mc\cosh\chi & mc\sinh\chi & mc\sinh\chi \\ \times\sin\theta\sin\phi & \times\cos\theta\sin\phi & \times\cos\theta\sin\phi & \times\sin\theta\cos\phi \\ \sinh\chi\cos\theta & mc\cosh\chi\cos\theta & -mc\sinh\chi\sin\theta & 0 \end{bmatrix} \qquad (121)$$

Hence

$$\eta_{\mu\nu} dp^{\mu} dp^{\nu} = \begin{bmatrix} d(mc) & d\chi & d\theta & d\phi \end{bmatrix} A^{\dagger} A \begin{bmatrix} d(mc) \\ d\chi \\ d\theta \\ d\phi \end{bmatrix} \qquad (122)$$

where

$$A^{\dagger} = \begin{bmatrix} -\cosh\chi & \begin{matrix}\sinh\chi \\ \times\sin\theta\cos\phi\end{matrix} & \begin{matrix}\sinh\chi \\ \times\sin\theta\sin\phi\end{matrix} & \sinh\chi\cos\theta \\ -mc\sinh\chi & \begin{matrix}mc\cosh\chi \\ \times\sin\theta\cos\phi\end{matrix} & \begin{matrix}mc\cosh\chi \\ \times\cos\theta\sin\phi\end{matrix} & mc\cosh\chi\cos\theta \\ 0 & \begin{matrix}mc\sinh\chi \\ \times\cos\theta\cos\phi\end{matrix} & \begin{matrix}mc\sinh\chi \\ \times\cos\theta\sin\phi\end{matrix} & -mc\sinh\chi\sin\theta \\ 0 & \begin{matrix}-mc\sinh\chi \\ \times\sin\theta\cos\phi\end{matrix} & \begin{matrix}mc\sinh\chi \\ \times\sin\theta\cos\phi\end{matrix} & 0 \end{bmatrix} \quad (123)$$

On matrix multiplication we obtain:

$$\eta_{\mu\nu}dp^{\mu}dp^{\nu} = -[d(mc)]^2 + (d\chi)^2 + \sinh^2\chi[(d\theta)^2 + \sin^2\theta(d\phi)^2] \quad (124)$$

$$A^{\dagger}A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \sinh^2\chi & 0 \\ 0 & 0 & 0 & \sinh^2\chi\sin^2\theta \end{bmatrix}$$