Fourier Transforms

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REPUBLIC OF SOUTH AFRICA

NRF

meerkat
karoo array telescope

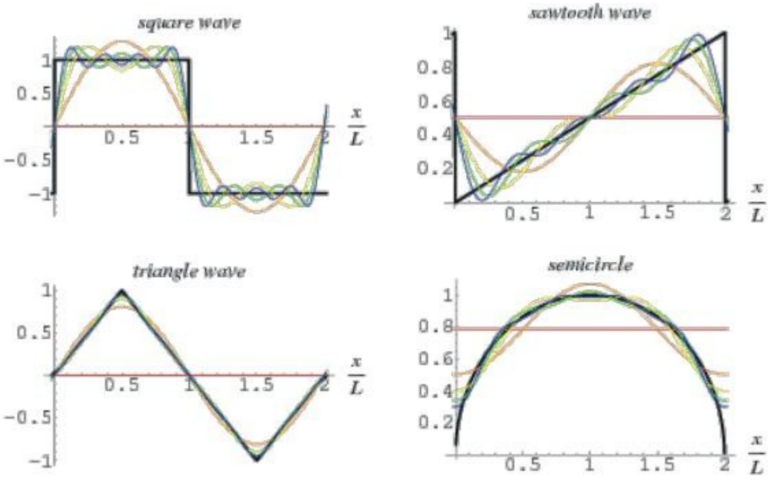
HartRAO

SKA
SQUARE KILOMETRE ARRAY

Who What When

Joseph Fourier was a mathematician from around the time of the French revolution

- **He realized that periodic functions could be decomposed into additions of simple sinusoids (sine and cosine functions) with multiples of the period**
- **The more you use, the better the approximation**



Why does it work?

Sine and cosine function form an orthonormal set if used from 0 to 2π (or -π to +π)

- $$\int_0^{2\pi} \cos(n\theta)\sin(n\theta)\,d\theta = 0$$

$$\int_0^{2\pi} \cos(n\theta)\cos(m\theta)\,d\theta = 0 \quad \text{if } m \neq n$$

$$\int_0^{2\pi} \sin(n\theta)\sin(m\theta)\,d\theta = 0 \quad \text{if } m \neq n$$

However...

$$\int_0^{2\pi} \cos(n\theta)\cos(n\theta)\,d\theta = \pi$$

$$\int_0^{2\pi} \sin(n\theta)\sin(n\theta)\,d\theta = \pi$$

How does that look

-

Mathematics

- We can treat this as a series of base vectors in an infinite dimensional space describing functions (the integral replaces the dot product)
- If we use the information that $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ it behaves nicely
- some arbitrary function $f(t) = A_1 e^{i\omega t} + A_2 e^{2i\omega t} + A_3 e^{3i\omega t} \ldots$ where A_1, A_2, A_3 are the complex fourier coefficients
-

$$A_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{i\omega nt} dt$$

Mathematics

- With a limited number of coefficients we can approximate any waveform (but more is better)
  - Higher order coefficients always get smaller
- Cosine components are the Real part and Sine components are the Imaginary part of the complex number
  - Any arbitrary functions can be described as an even and an odd part
  - Odd functions are the Sine parts, and Even functions cosine parts

$$f(x) = \frac{f(+x) + f(-x)}{2} + \frac{f(+x) - f(-x)}{2} = f_{even}(x) + f_{odd}(x)$$

What if it isn't periodic?

- Then we get a continuous function instead of a series of discrete coefficients
- This is the Fourier Transform

$$f(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(\omega)\, e^{i\omega t}\, dt$$

Which has its counterpart

$$g(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, d\omega$$

- Depending on domains you don't need to start from -∞ ; if you start at 0 you drop the “2” in the denominator

Some theorems

- If a function is symmetric about the origin its FT is cosines only (real)
- If a function is antisymmetric abou the origin its FT is sines only (imaginary)
    - same as a cosine with a $\pi/2$ (90º)phase shift
- If we move off from the origin we get a phase slope

$$f(t+a)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(\omega)\, e^{i\omega t}\, dt\, .\, e^{i a\omega t}$$

What You Need To Know

If F(u) is the fourier transform of f(x)

- **The inverse of a fourier transform is just another fourier transform (with -*i* in stead of +*i*) so they behave EXACTLY the same**
- **Wide in f is narrow in F (and vice versa)**
- **Edgy in f is ripply in F**
- **A sine (or cosine) wave in f is a point in F and the shorter the period the further from the origin**
- **A point at the origin f is an infinitely long line in F (consider it as a cosine of infinite period)**
- **An infinite series of equally spaced points in F is an equally spaced infinite series of points in f (but see above..)**

What You Need To Know

If F(u) is the Fourier transform of f(x) and G(u) is the Fourier transform of g(x) then:

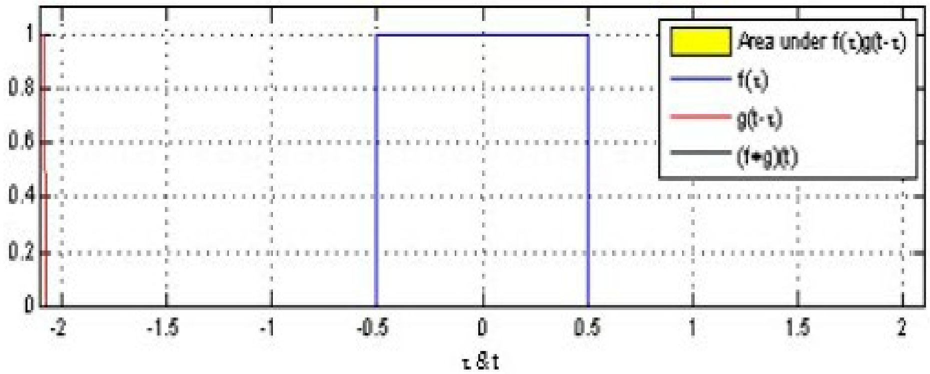
Scale f(ax) = $$\frac{1}{|a|}F\left(\frac{u}{a}\right)$$

Transform (f+g) gives F+G (they add), and scale

Transform (f.g) gives F convolved with G (F⊗G) they do not multiply!

$$\int_{-\infty}^{\infty} F(v)\,G(v-u)\,dv$$

Convolution ?



What You Need To Know

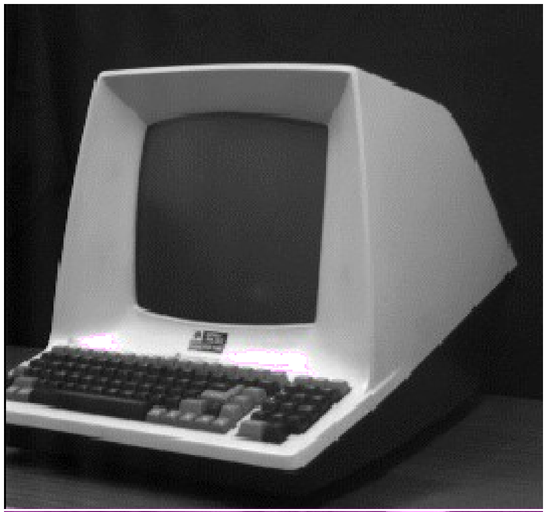
- **A phase flip in one domain inverts the other domain**
- **An offset in one domain is an phase slope in the other domain**

$$f(t+a)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(\omega)\,e^{i\omega t}\,dt\,.\,e^{ia\omega t}$$

- **Phase information tells you <u>where</u> stuff is, amplitude how <u>strong</u> it is**

Examples

- Taken from web; Fourier transform two images



Phase from **face** and amplitude from **terminal.** Transform back again...

A phase-only image (amplitude =1) is also recognisable

Why do we need this?

FTs turns up in any wave type phenomenon

- Natural modes of vibration
- Quantum Mechanics
- Electromagnetism -diffraction is a BIG effect at long wavelengths
- Heat diffusion
- X-ray Crystallography
- Image processing (we use EM waves to measure them)
- Sound processing (we use sound waves to hear)
- Tides
- Digital electronics

Examples in Radio Astronomy

- Time transforms into frequency, so an the FT of an autocorrelation of a signal with a version stepped in time gives a spectrum

  - Wiener–Khintchine theorem

- Angles (on the sky) transform into baseline lengths in wavelength (in the (u,v) plane)

  - Van Cittert–Zernike theorem

- If you lose the long baseline (high frequency) components you lose the fine resolution structure in angle ('high notes') & if you lose short baseline components (low frequency) you lose the large scale structure in angles ('bass notes')
- The total intensity of the whole field is contained at a baseline spacing of zero
- The transform of a (dish surface and illumination and blockage) is beam pattern of the dish on the sky

Wiener-Khinchin theorem

Power spectral density of a wide-sense-stationary random process is the Fourier transform of the corresponding autocorrelation function

- Einstein noticed this!

- Most radio spectrometers (and some spectrum analysers) work this way instead of using filters, as well as some infrared spectrometers
- This is also used in a LOT of digital systems

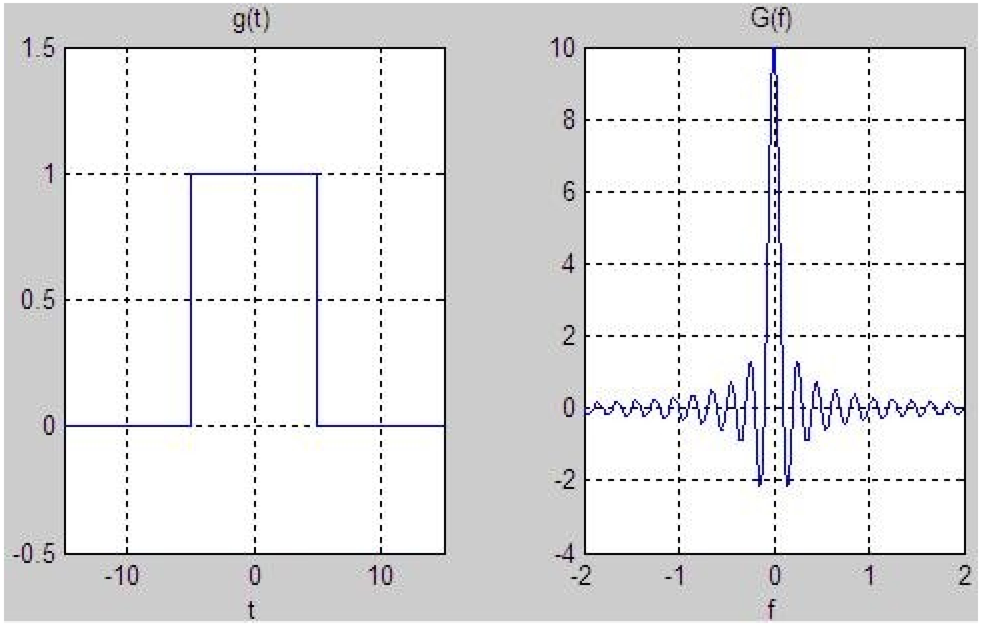
Van Cittert-Zernike theorem

- **Fourier transform of the mutual coherence function of a distant, incoherent source is equal to its complex visibility**

- This is how radio astronomers make images of fields, since our baselines sample the mutual coherence function if we make correlators from the electric field vectors.

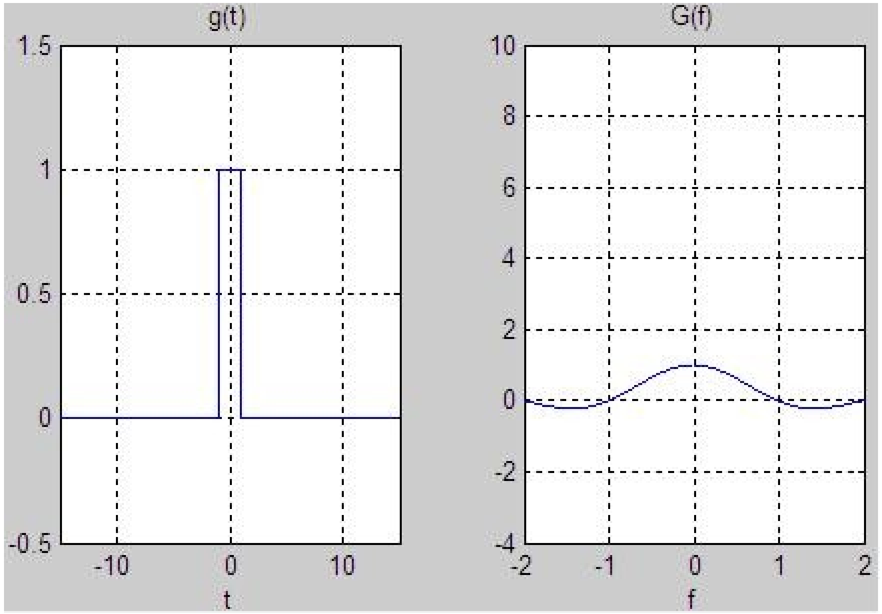
Some examples

Top hat transforms to sinc function



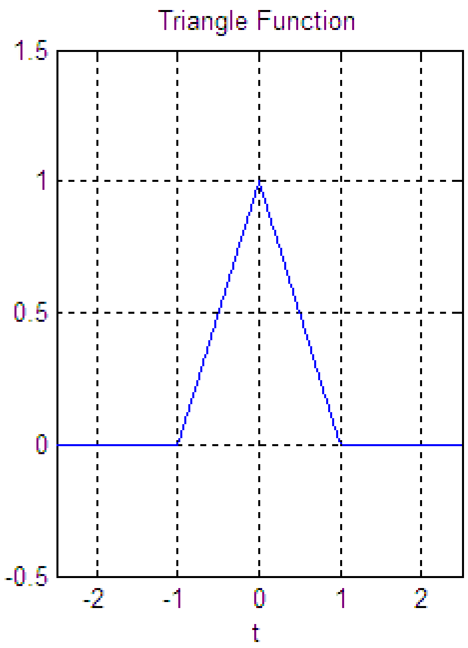
Narrow-Wide

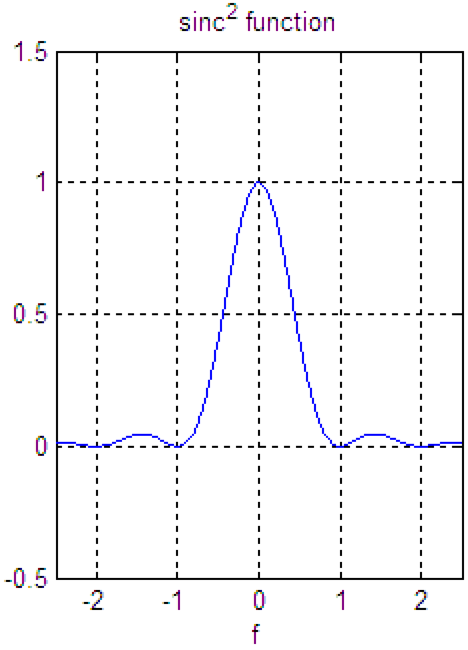
A narrower top hat transforms to a wider sinc



Triangle?

It can be seen as the convolution of a pair of top hats





Gibbs Phenomenon

What do we get with successive terms for a square wave

Smooth it Gibbs

By applying a smoothing function...

See demos

fourier.py

fourier-tophat.py

2D version

- **For example on sky from measured correlations V with north-south "u" and east west "v" components...**

$$I(l,m)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} V(u,v)\, e^{2\pi i ul}\, e^{2\pi i vm}\, du\, dv$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} V(u,v)\, e^{2\pi i(ul+vm)}\, du\, dv$$

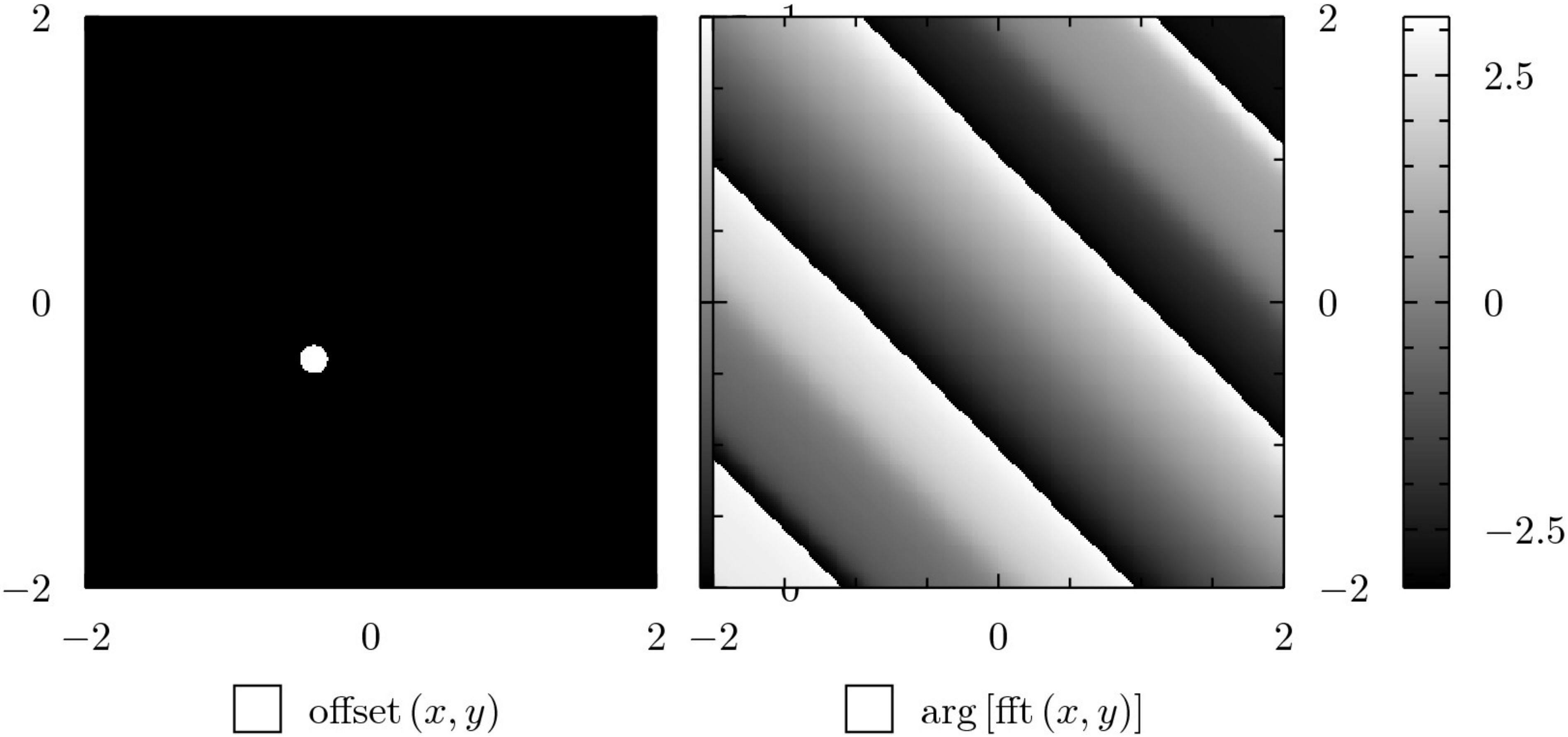
2D tophat

A narrower top hat transforms to a wider sinc

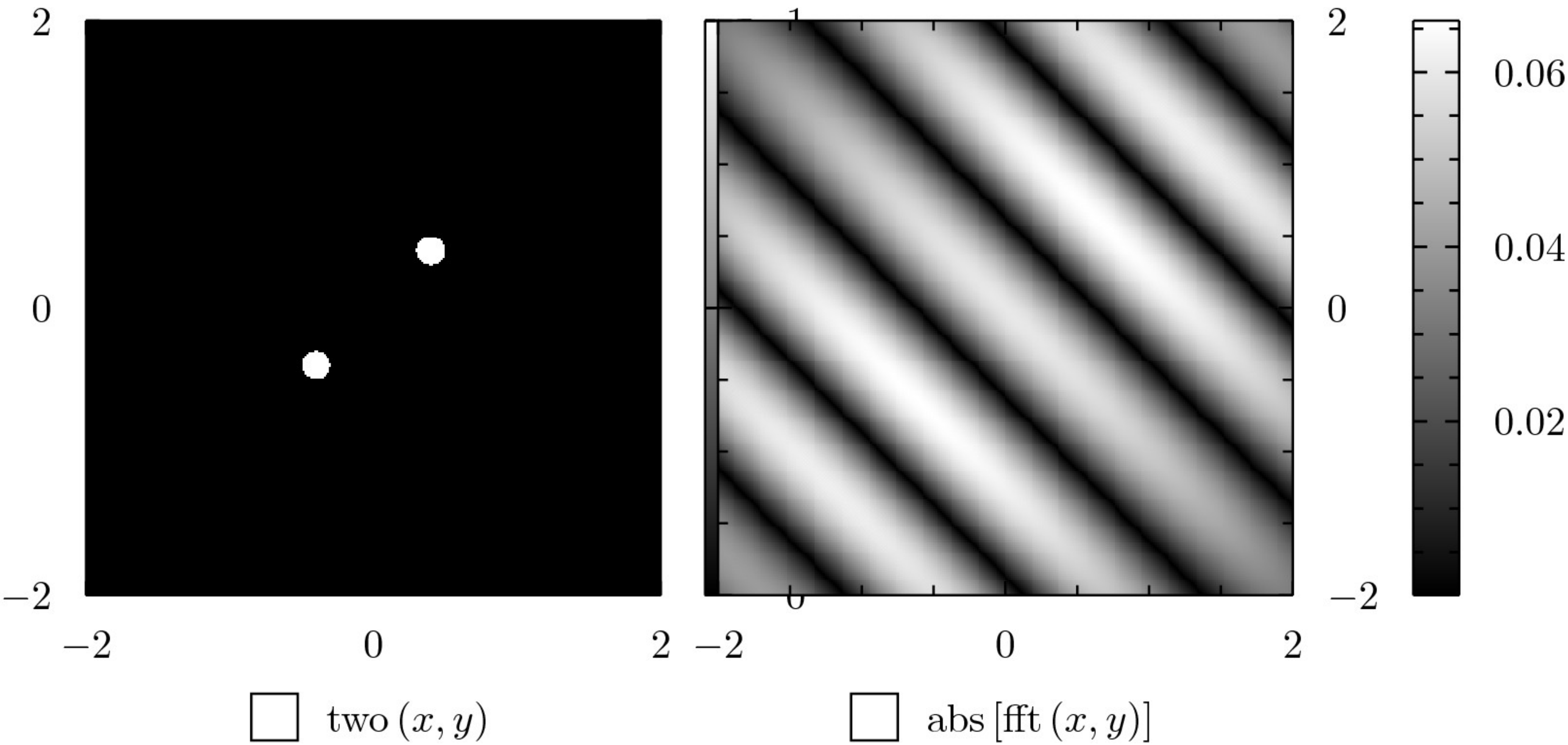
2D Gaussian

Wide Gaussian

Offsets -phase slope



Two components



Piece of cake?

All you need to do is make measurements over infinite area!

- Well if you accept up to a finite resolution you can accept a full coverage up to a certain value
- If we don't need to image the whole sky we can get away without a full sampling
- We actually measure discrete values rather than contiuous ones so it is more practical to take the discrete case

Piece of cake?

All you need to do is make measurements over infinite area!

- Well if you accept up to a finite resolution you can accept a full coverage up to a certain value
- If we don't need to image the *whole* sky we can get away without a full sampling, with a few assumptions...
- We actually measure discrete values rather than contiuous ones so it is more practical to take the discrete case

Matrix version

- As we can take each fourier component as a set of orthogonal "vectors" we can handle them in a matrix formalism with a matrix for discrete data

If we set $w = e^{i\omega t}$

then the FT looks like

$$\begin{bmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & w & w^2 & w^3 & \dots & w^{n-1} \\ 1 & w^2 & w^4 & w^6 & \dots & w^{2(n-1)} \\ \vdots & & & & \ddots & \vdots \\ 1 & w^{(n-1)} & w^{2(n-1)} & w^{3(n-1)} & \dots & w^{(n-1)(n-1)} \end{bmatrix}$$

- This works excellently for regularly gridded data, and using the symmetries inherent in this people have developed a **Fast Fourier Transform**
  - works best with a power of 2 (256, 512, 1024, 2048...)

examle - 4x4 transform

$$w^4 = 1 \quad \text{so} \quad w = e^{2\pi/4} = i$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$$

to normalize divide by $\frac{1}{\sqrt{4}} = \frac{1}{2}$

this is equivalent to the $\frac{1}{\sqrt{2\pi}}$ for the continuous case

Footnote

- We actually receive voltages from dishes over a time series and we want correlations in a number of frequency bands
  - we can crosscorrelate (“X”) and Fourier transform the time data to frequency band (“F”) - an “XF” correlator
  - or split to frequency bands and then crosscorrelate - a “FX” correlator
- In either case we get 3 dimensional data out (with 4 polarization products)