High Energy Astrophysics & Pulsars Prof: Pieter Meintjes

Wendyam Blaise Tapsoba, *University of Cape Town (twblaise1er@gmail.com)*

I - Decoding the cosmos

⊙ - Chap 1 : Defining the signal - ⊙

• The **luminosity** of a object can be expressed in function of the spectral luminosity per unit wavelength L_λ or per unit frequency L_ν by :

$$dL = L_\lambda\, d\lambda = L_\nu\, d\nu \Rightarrow L = \int_0^\infty L_\lambda\, d\lambda = \int_\infty^0 L_\nu\, d\nu$$

$$\lambda = \frac{c}{\nu} \implies d\lambda = \frac{-c}{\nu^2}\, d\nu$$

$$\int_0^\infty L_\lambda\, d\lambda = \int_0^\infty (\frac{-c}{\nu^2}) L_\lambda\, d\nu = \int_\infty^0 (\frac{c}{\nu^2}) L_\lambda\, d\nu$$

Therefore :

$$L_\nu = L_\lambda \frac{c}{\nu^2} \quad \implies \quad L_\lambda = \frac{\nu^2}{c} L_\nu$$

,

• The **spectral flux density**, or just flux density $(ergs^{-1}cm^{-2}Hz^{-1})1$ is the flux per unit spectral bandwidth, either frequency or wavelength, respectively:

$$dL_\nu = f_\nu dA \quad dL_\lambda = f_\lambda dA$$

$$df = f_\nu d\nu \quad df = f_\lambda d\lambda \quad \boxed{f = f_\nu \Delta\nu} \boxed{dL = f dA}$$

Then the luminosity, L, can be found from its flux, f $(ergs^{-1}cm^{-2})$, via : $\boxed{L = \int f dA}$

For a isotropic radiation $L_{isotropic} = 4\,\pi\, r^2\, f$ and for uniform radiation $L_{uniform} = r^2\,\Omega\, f$. Where r is the distance from the centre of the source to the position at which the flux has been determined.

• A detector pointed directly at a uniform intensity source in the sky of small solid angle, Ω, would measure a flux, $f = \int_\Omega I \cos\theta d\Omega \approx I\Omega$.

• The astrophysical flux at the **surface of an object** (e.g. a star) whose radiation is escaping freely at all angles outwards, can be calculated by integrating in spherical coordinates.

$$F = \int I \cos\theta d\Omega = \int_0^{2\pi} \int_0^{\pi/2} I \cos\theta \sin\theta d\theta d\phi = \pi I$$

• For stars, we now define the astrophysical flux, F, to be the flux at the surface of the star,

$$L_* = 4\pi R_*^2 F = 4\pi r^2 f \implies f = (\frac{R_*}{r})F$$

Where L_* is the star's luminosity and R_* is its radius.

• The solid angle Ω is : $\Omega = \frac{\pi}{4}\theta^2 = \frac{\pi}{4}(\frac{d}{r})^2$. Where d is the linear diameter. **Ex:** with radiation beamed uniformly into a circular cone of solid angle, Ω. $\boxed{A = \pi R_*^2 = r_{mars-*}^2 \Omega}$. $\boxed{\Omega = \pi(\frac{R_*}{r_{mars-*}})^2}$

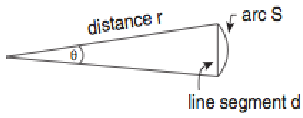

Figure 1: *arc* $s = r\theta$. *If* $d \ll r$, $s = d$, *hence* $\theta = \frac{d}{r}$

N.B: If the source in the sky is elliptical rather than circular, in projection, then, $\boxed{\Omega = \frac{\pi}{4}\theta_1\,\theta_2}$

⊙ - Chap 8 : Continuum emission - ⊙

• The root-mean-square speed, v_{rms}, corresponding kinetic energy, $E_k = \frac{3}{2}\,kT = \frac{1}{2}mv_{rms}^2$, $v_{rms} = \sqrt{\frac{3kT}{m}}$

• The spectrum, in the absence of a background source, $I_\nu = B_\nu(1 - e^{-\tau_\nu})$. $B_\nu(T) = \frac{2\,h\,\nu^3}{c^2}\frac{1}{e^{\frac{h\,\nu}{k\,T}} - 1}$

• The emission measure: $\epsilon M = \int n_e^2\, dl \approx n_e^2\, l$

• For $h\nu \ll kT_e$ $1 - e^{-\frac{h\nu}{kT}} \approx \frac{h\nu}{kT}$. (Opti thick).

• The **ionized hydrogen** mass M_{HII} and the total gas mass, M_g are related by : $M_{HII} = X\, M_g$. Where X is the mass fraction of hydrogen : $X = 0.707$ for Solar abundance.

The radius, R_s of the *Strömgren sphere* assuming that the effect of dust are negligible is:

. $\boxed{U = R_s\, n_e^{2/3}}$

• The specific intensity : $I_\nu = \frac{2\,h\nu^3}{c^2}\frac{1}{(e^{\frac{h\nu}{kT_B}}-1)}$.

For $h\nu \ll kT_B$:

$$I_\nu = \frac{2h\nu^3}{c^2}\frac{1}{(e^{\frac{h\nu}{kT_B}}-1)} \approx \frac{2h\nu^3}{c^2}\frac{1}{(1+\frac{h\nu}{kT_B}+\cdots-1)}$$

$$\boxed{I_\nu = \frac{2\,\nu^2\,kT_B}{c^2}}$$

The flux density of the star :

$$F_\nu = \pi I_\nu = \pi\frac{2\,\nu^2\,kT_B}{c^2} \Rightarrow \boxed{f_\nu = (\frac{R_*}{r})^2\pi\frac{2\,\nu^2\,kT_B}{c^2}}$$

.

• The relativistic energy of the photon given by : $E_p = \gamma\, m_p\, c^2$. The relativistic gyroradius (kpc) of a proton : $r = \gamma r_0 = \frac{\gamma m_0 c^2}{eB}$

• The energy spectral index: $\Gamma = 1 - 2\alpha$. The observed spectral index $\alpha = -1$

$$I_\nu = S_\nu(1-e^{-\tau_\nu}) \quad (all\ \ \tau_\nu)$$

$$I_\nu = S_\nu \propto \nu^{\frac{5}{2}} \quad (\tau_\nu \gg 1)\ \ Opti\ \ thick$$

$$I_\nu = j_\nu \propto \nu^{-\alpha} \quad (\tau_\nu \ll 1)\ \ Opti\ \ thin$$

.

Do problem 8.13 of Decoding the Cosmos.

. . . .

. **End tutorial sum**

. . . .

II - Radiative Processes In High E Ast

⊙ - Chap 1 : Some Fundamental definitions - ⊙

• **Emission plus absorption :** In this case it is convenient to introduce the optical depth τ_ν:

. $d\tau_\nu = \alpha_\nu ds = n\sigma_\nu ds.$

σ_ν: cross section of the absorbing process, and n density of "absorbers".

The transport equation then becomes:

$$\frac{dI_\nu}{\alpha_\nu ds} = -I_\nu + \frac{j_\nu}{\alpha_\nu} \implies \frac{dI_\nu}{d\tau_\nu} = -I_\nu + \frac{j_\nu}{\alpha_\nu}, \quad S_\nu = \frac{j_\nu}{\alpha_\nu}$$

$$I_\nu(\tau_\nu) = I_{\nu,0}e^{-\tau_\nu} + \int_0^{\tau_\nu} e^{-(\tau_\nu-\tau'_\nu)}S_\nu(\tau'_\nu)d\tau'_\nu$$

If the **source function** S_ν is constant :

$$I_\nu(\tau_\nu) = I_{\nu,0}e^{-\tau_\nu} + S_\nu(1-e^{-\tau_\nu})$$

If $I_{\nu,0} = 0$, $I_\nu(\tau_\nu) = \frac{j_\nu}{\alpha_\nu}(1-e^{-\tau_\nu})$. Let $s = R$

$$I_\nu(\tau_\nu) = \frac{j_\nu R}{\alpha_\nu R}(1-e^{-\tau_\nu}) = j_\nu(\frac{1-e^{-\tau_\nu}}{\tau_\nu})$$

∘ Source optically thin $(\tau_\nu \ll 1)$, we have $1 - e^{-\tau_\nu} \to 1 - 1 + \tau_\nu$. Therefore :

$$I_\nu(\tau_\nu) = j_\nu R \quad (\tau_\nu \ll 1)$$

∘ Source optically thick $(\tau_\nu \gg 1)$

$$I_\nu(\tau_\nu) = \frac{j_\nu R}{\tau_\nu} \quad (\tau_\nu \gg 1)$$

The above equation explicitly shows that the intensity we see from a thick source comes from a layer of width R/τ_ν, i.e. the layer that is optically thin. In other words we always collect radiation from a layer of the source, down to the depth at which the radiation can escape without being absorbed ($\tau_{layer} = 1$).

• The mean free path (for a photon) is the average distance l travelled by a photon without interacting. It corresponds to a distance for which $\tau_\nu = 1$: $\tau_\nu = 1 \to \sigma_\nu n l_\nu = 1 \to l_\nu = \frac{1}{n\sigma_\nu}$.

If a source has radius R and total optical depth $\tau_\nu > 1$, we have : $l_\nu = \frac{1}{n\sigma_\nu} = \frac{R}{\sigma_\nu n R} = \frac{R}{\tau_\nu}$

• **Total emitted power: Larmor formula**

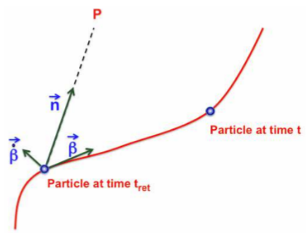

Figure 2: *A charge is moving along a trajectory.*

The electric and magnetic fields produced by a moving charge is:

$$\vec{E}(\vec{r},t) = [\frac{q}{k^2R^2}\frac{\vec{n}-\vec{\beta}}{\gamma^2}]_{t_{ret}} + \frac{q}{ck^3R}\{\vec{n}\times[(\vec{n}-\vec{\beta})\times\dot{\vec{\beta}}]\}_{t_{ret}}$$

$$\vec{B}(\vec{r},t) = \vec{n}\times\vec{E}$$

Where: $t_{ret} = t - \frac{R(t_{ret})}{c}$, $k = 1 - \vec{n}.\vec{\beta}$, $\vec{\beta} = \frac{\vec{v}}{c}$

. $\gamma = (1-\beta^2)^{-1/2}$

we specialize to the non relativistic case, consider only the radiative field, and set:

$\vec{\beta} \ll 1$, $k = 1 - \vec{n}.\vec{\beta} \to 0$, $\vec{n} - \vec{\beta} \to \vec{n}$

To calculate the power per unit solid angle carried by this electromagnetic field let consider the **Poynting vector** $\vec{S}$ (in cgs units):

$$\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B} = \frac{c}{4\pi}|\vec{E}|^2\vec{n} = \frac{c}{4\pi}\frac{q^2}{c^2R^2}|\vec{n}\times(\vec{n}\times\dot{\vec{\beta}})|^2_{t_{ret}}$$

The power crossing a surface $dA = R^2 d\Omega$ is :

$$dP = SdA = SR^2 d\Omega \rightarrow \frac{dP}{d\Omega} = sR^2$$

Therefore :

$$\frac{dP}{d\Omega} = \frac{q^2}{4\pi c}|\vec{n} \times (\vec{n} \times \dot{\vec{\beta}})|^2_{t_{ret}} = \frac{q^2}{4\pi c}\dot{\vec{\beta}} \sin^2\theta$$

$$\boxed{\frac{dP}{d\Omega} = \frac{q^2}{4\pi c^3} a^2 \sin^2\theta} \quad \textbf{Larmor formula}$$

The **total power P :**

$$P = int\frac{dP}{d\Omega}d\Omega = \frac{q^2 a^2}{4\pi c^3}\int_{-1}^{1} \sin^2\theta d(d\cos\theta) = \frac{2q^2}{3c^3}a^2$$

⊙ - Chap 2 : Bremsstrahlung and black body - ⊙

• Bremsstrahlung

We will consider an electronproton plasma.
b: impact parameter; v: velocity of the electron;
n_e: number density of the electrons;
n_p: number density of the protons;
T: temperature of the plasma: $mv^2 \sim kT \rightarrow v \sim (kT/m)^{1/2}$

○ Let consider e^- passing close to the photon. Caracteristic **(or interaction) time** : $\tau \approx \frac{b}{v}$
○ During the interaction we consider the acceleration to be constant : $m_e a \approx F_e \approx \frac{e^2}{b^2} \rightarrow a \approx \frac{e^2}{m_e b^2}$

○ From the Larmor formula we get :

$$P = \frac{2q^2}{3c^3}a^2 \approx \frac{e^2}{c^3}\frac{e^4}{m_e^2 c^3 b^4} = \frac{e^6}{m_e^2 c^3 b^4}$$

○ Since there is a characteristic time, there is also a characteristic frequency : $\omega \approx \frac{1}{\tau} = \frac{v}{b}$

$$P(\omega) = \frac{P}{\omega} = \frac{[\frac{e^6}{m_e^2 c^3 b^4}]}{\omega} = [\frac{e^6}{m_e^2 c^3 b^4}]\frac{v}{b} \approx \frac{e^6}{m_e^2 c^3 b^3 v}$$

○ We can estimate the impact factor b from the density of protons (targets) : $b \approx n_p^{-1/3} \rightarrow b^3 = \frac{1}{n_p}$

○ The emissivity $j(\omega)$ will be the *power emitted by a single electron* mul- tiplied by the number density of electrons. If the emission is isotropic we will have to divide by 4π since the emissivity is directional : $j(\nu) \rightarrow ergcm^{-3}s^{-1}Hz^{-1}sr^{-1}$

$$j(\omega) = \frac{n_e}{4\pi}P(\omega) = \frac{n_e}{4\pi}\frac{e^6}{m_e^2 c^3 b^3 \nu} = \frac{n_e}{4\pi}\frac{1}{b^3}\frac{e^6}{m_e^2 c^3}\frac{1}{v}$$

$$j(\omega) = \frac{n_e}{4\pi}n_p\frac{e^6}{m_e^2 c^3}(\frac{kT}{m_e})^{-1/2} = \frac{n_e n_p}{4\pi}\frac{e^6}{m_e^2 c^3}(\frac{m_e}{kT})^{1/2}$$

○ **The total emissivity** j depend upon $\hbar\omega_{max} = kT$. This means that an electron cannot emit a photon of energy larger than $\hbar w_{max}$, i.e limited by the average energy kT. Hence by doing this we ignore electrons with energies greater than kT.

$$j = \int_0^{w_{max}} j(\omega)d\omega \approx j(\omega)\ \omega_{max}$$

$$j = \frac{n_e n_p}{4\pi}\frac{e^6}{m_e^2 c^3}(\frac{m_e}{kT})^{1/2}\ \omega_{max} = \frac{n_e n_p}{4\pi}\frac{e^6}{m_e^2 c^3}(\frac{m_e}{kT})^{1/2}\ (\frac{kT}{\hbar})$$

$$j = \frac{n_e n_p}{4\pi}\frac{e^6}{m_e^2 c^3}\frac{(m_e kT)^{1/2}}{\hbar}$$

○

$$< \frac{dE}{dvdtd\omega} >= \frac{\int_{v_{min}}^{\infty} j_\omega(v)dP(v)}{\int_o^\infty dP(v)}$$

$$< \frac{dE}{dvdtd\omega} >= \frac{\frac{16\pi e^6}{3\sqrt{3}m_e^2 c^3}n_e n_i Z^2 g_{ff}\int_{v_{min}}^\infty \frac{1}{v}\ e^{-\frac{m_e v^2}{2kT}}d^3v}{\int_0^\infty e^{\frac{-mv^2}{2kT}}d^3v}$$

$$< \frac{dE}{dvdtd\omega} >= \frac{\frac{16\pi e^6}{3\sqrt{3}m_e^2 c^3}n_e n_i Z^2 g_{ff}\int_{v_{min}}^\infty v\ e^{-\frac{m_e v^2}{2kT}}dv}{\int_0^\infty v^2 e^{\frac{-m_e v^2}{2kT}}dv}$$

$$\int_0^\infty v^2 e^{\frac{-m_e v^2}{2kT}}dv = [\ \frac{\sqrt{\pi}}{4(\frac{m_e}{2kT})^{3/2}}\ ]$$

$$\int_{v_{min}}^\infty v\ e^{-\frac{m_e v^2}{2kT}}dv = \frac{kT}{m}e^{\frac{-m_e v_{min}^2}{2kT}}$$

$$< \frac{dE}{dvdtd\omega} >= \frac{32\sqrt{\pi}e^6 n_e n_i Z^2 g_{ff}(\frac{m_e}{2kT})^{1/2}e^{-\frac{m_e v_{min}^2}{2kT}}}{3\sqrt{3}m_e^2 c^3}$$

Using $v_{min}^2 = \frac{2h\nu}{m_e}$

$$\boxed{< \frac{dE}{dvdtd\omega} >= \frac{32\sqrt{\pi}e^6 n_e n_i Z^2 g_{ff}(\frac{m_e}{6kT})^{1/2}e^{-\frac{h\nu}{kT}}}{3m_e^2 c^3}}$$

$j_\omega = \frac{dj}{d\omega} = \frac{dj}{2\pi d\nu}$; with $\omega = 2\pi\nu \rightarrow d\omega = 2\pi d\nu$

$j_\nu = \frac{dj}{d\nu} = 2\pi j_\omega,\ j_\nu =< \frac{dE}{dvdtd\nu} >,\ j_\omega =< \frac{dE}{dvdtd\omega} >$

Then $j_\nu =< \frac{dE}{dvdtd\nu} >= 2\pi < \frac{dE}{dvdtd\omega} >$. Hence

$$j_\nu = \frac{32\pi}{3}\frac{n_e n_i e^6 Z^2 g_{ff}}{3m_e^2 c^3}(\frac{2\pi m}{3kT})^{1/2}\ e^{-\frac{h\nu}{kT}}$$

$$\boxed{j_\nu(\Omega) = \frac{j_\nu}{4\pi} = \frac{8}{3}(\frac{2\pi}{3})^{1/2}\frac{n_e n_i e^6 Z^2}{3m_e^2 c^3}(\frac{m_e}{kT})^{1/2}\ e^{-\frac{h\nu}{kT}}g_{ff}}$$

$$j(\Omega) = \int_0^\infty j_\nu(\Omega)d\nu \quad n_i = n_p;\ \ j_\nu(\Omega) = j(\nu);\ j = j(\Omega)$$

$$\boxed{j(\Omega) = \frac{4}{3\pi}(\frac{2\pi}{3})^{1/2}\frac{n_e n_i e^6 Z^2}{3m_e^2 c^3}\frac{(m_e KT)^{1/2}}{\hbar}g_{ff}} \quad \hbar = \frac{h}{2\pi}$$

• Free – free absorption

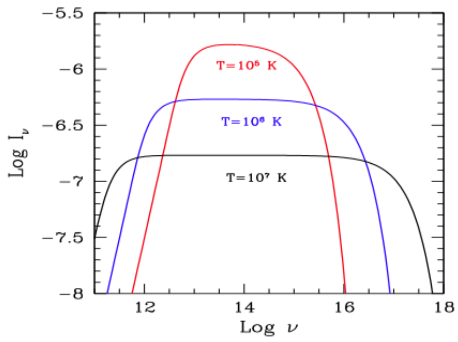

Figure 3: *Changes of the Bremsstrahlung intensity with temperature.*

The three spectra correspond to different temperatures. Note that for smaller temperatures the thin part of I_ν is larger ($I_\nu \propto T^{-1/2}$). On the other hand, at larger T the spectrum extends to larger frequencies, making the frequency integrated intensity to be larger for larger T ($I_\nu \propto T^{1/2}$). Note also the self-absorbed part, whose slope is proportional to ν^2. This part ends when the optical depth $\tau \propto \alpha_\nu R \sim 1$.

• From bremsstrahlung to black body

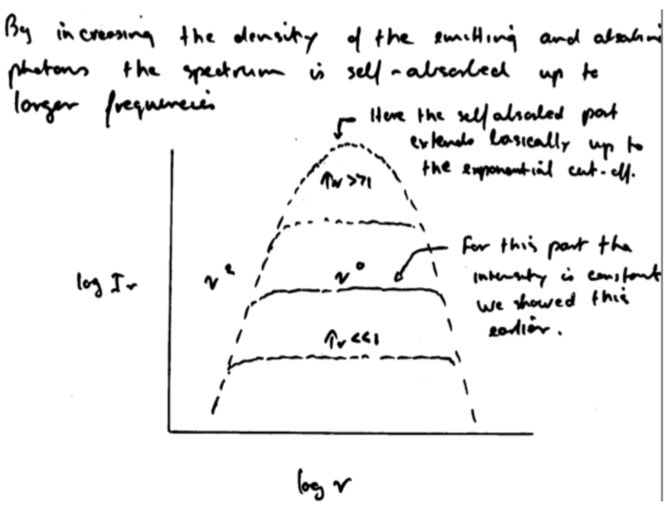

Figure 4: *The bremsstrahlung intensity becomes more and more self-absorbed as the density increases, until it becomes a blackbody. At this point increasing the density does not increase the intensity any longer. This is because we receive radiation from a layer of unity optical depth. The width of this layer decreases as we increase the densities, but the emissivity increases, so that:* $I_\nu = \frac{j_\nu R}{\tau_\nu} \propto \frac{n_e n_p R}{n_e n_p R} \rightarrow$ *constant* $(\tau_\nu \gg 1)$

• Black body

The black body intensity is given by : $B_\nu = \frac{2\,h\nu^3}{c^2}\frac{1}{(e^{\frac{h\nu}{kT}}-1)}$.

○ $I_\nu^{RJ} = \frac{2\nu^2\,kT}{c^2} \quad h\nu \ll kT \Rightarrow e^{\frac{h\nu}{kT}} - 1 \rightarrow \frac{h\nu}{kT}$

○ $I_\nu^{Wien} = \frac{sh\nu^3}{c^2} e^{-\frac{h\nu}{kT}}$; $h\nu \gg kT \Rightarrow e^{\frac{h\nu}{kT}} - 1 \rightarrow e^{\frac{h\nu}{kT}}$

⊙ - Chap 3 : Beaming - ⊙

• The moving bar

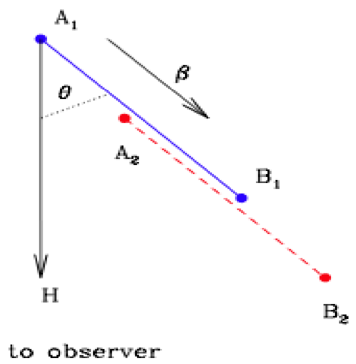

Figure 5

Let us consider a moving bar, of proper dimension l', moving in the direction of its length at velocity βc and at an angle θ with respect to the line of sight. The length of the bar in the frame K (according to relativity "without photons") is $l = l'/\Gamma$. The photon emitted in A_1 reaches the point H in the time interval Δt_e. After Δt_e the extreme B_1 has reached the position B_2, and by this time, photons emitted by the other extreme of the bar can reach the observer simultaneously with the photons emitted by A_1, since the travel paths are equal. The length $B_1B_2 = \beta c \Delta t_e$, while $A_1H = c\Delta t_e$. Therefore:

$$A_1H = A_1B_2 \cos\theta \rightarrow \Delta t_e = \frac{l' \cos\theta}{c\Gamma(1-\beta\cos\theta)}$$

The length A_1B_2 is then given by :

$$A_1B_2 = \frac{A_1H}{\cos\theta} = \frac{l'}{\Gamma(1-\beta\cos\theta)} = \delta l'$$

In a real picture, we would see the projection of A_1B_2, i.e.:

$$HB_2 = A_1B_2 \sin\theta = l' \frac{\sin\theta}{\Gamma(1-\beta\cos\theta)} = l'\delta\sin\theta$$

The observed length depends on the viewing angle, and reaches the maximum (equal to l') for $\cos\theta = \beta$.

- **Time - Frequency**

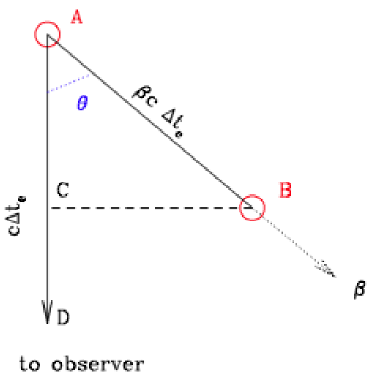

Figure 6

Consider a lamp moving with velocity $v = \beta c$ at an angle θ from the line of sight. In K', the lamp remains on for a time $\Delta t'_e$. According to special relativity ("without photons") the measured time in frame K should be $\Delta t_e = \Gamma \Delta t'_e$ (time dilation). However, if we use photons to measure the time interval, we once again must consider that the first and the last photons have been emitted in different location, and their travel path lengths are different. To find out Δt_a, the time interval between the arrival of the first and last photon. The first photon is emitted in A, the last in B. If these points are measured in frame K, then the path AB is :

$$AB = \beta c \Delta t_e = \Gamma \beta c \Delta t'_e$$

While the lamp moved from A to B, the photon emitted when the lamp was in A has travelled a distance $AC = c\Delta t_e$, and is now in point D. Along the direction of the line of sight, the first and the last photons (the ones emitted in A and in B) are separated by CD. The corresponding time interval, CD/c, is the interval of time Δt_a between the arrival of the first and the last photon:

$$\Delta t_a = \frac{CD}{c} = \frac{AD - AC}{c} = \Delta t_e - \beta \Delta t_e \cos\theta$$

$$= \Delta t_e(1 - \beta\cos\theta) = \Delta t'_e \Gamma (1 - \beta \cos\theta)$$

$$\boxed{\Delta t_a = \frac{\Delta t'_e}{\delta}}$$

If θ is small and the velocity is relativistic, then $\delta > 1$, and $\Delta t_a < \Delta t_s$, i.e. we measure a time contraction instead of time dilation. Note also that we recover the usual time dilation (i.e. $\Delta t_a = \Gamma \Delta t'_e$) if $\theta = 90^o$, because in this case all photons have to travel the same distance to reach us.

Since a frequency is the inverse of time, it will transform as : $\boxed{\nu = \nu'}$. It is because of this that the factor δ is called the relativistic Doppler factor.

$$\delta = \frac{1}{\Gamma(1 - \beta\cos\theta)}$$

- **Aberration**

Calling θ the angle between the direction of the emitted photon and the source velocity vector, we have:

$$\sin\theta = \frac{\sin\theta'}{\Gamma(1+\beta\cos\theta')}; \quad \sin\theta' = \frac{\sin\theta}{\Gamma(1-\beta\cos\theta)}$$

$$\cos\theta = \frac{\cos\theta' + \beta}{1+\beta\cos\theta'}; \quad \cos\theta' = \frac{\cos\theta - \beta}{1-\beta\cos\theta}$$

The definition of solid angle is:

$$d\Omega = \sin\theta d\theta d\phi = 2\pi \sin\theta d\theta$$

$$d\Omega = 2\pi [\frac{\sin\theta'}{\Gamma(1+\beta\cos\theta')}] \, [\frac{d\theta'}{\Gamma(1+\beta\cos\theta')}]$$

$$d\Omega = \frac{2\pi \ \sin\theta' d\sin\theta'}{[\Gamma(1+\beta\cos\theta')]^2}$$

But $d\Omega' = 2\pi \sin\theta' d\theta'$; $d\Omega = d\Omega' \frac{1}{[\Gamma(1+\beta\cos\theta')]^2}$

With $\cos\theta' = \frac{\cos\theta - \beta}{1-\beta\cos\theta}$, we get $\boxed{d\Omega = \frac{d\Omega'}{\delta^2}}$

- **Intensity**

The specific intensity has the unit of energy per unit surface, time, frequency and solid angle:

$$I_\nu = h\nu \, \frac{dN}{dt\, d\nu\, d\Omega\, dA}$$

$\Delta t = (1-\beta\cos\theta)\Gamma\Delta t' = \frac{\Delta t'}{\delta}$; $\nu = \frac{\nu'}{\Gamma(1-\beta\cos\theta)} = \delta\nu'$

The surface area stays constant since it is perpendicular to the direction of the light source $\Rightarrow dA = dA'$. In the comoving frame K' ($d\nu = \delta\, d\nu'$):

$$I'_{\nu'} = h\nu' \frac{dN'}{dt' d\nu' d\Omega' dA'}$$

$$I_\nu = h\delta\nu' \frac{dN'}{(\frac{dt'}{\delta})\delta d\nu' (\frac{d\Omega'}{\delta^2}) dA'} = \delta\, h\, \nu' \, \frac{dN'}{(\frac{1}{\delta^2})\, dt'\, d\nu'\, d\Omega'\, dA'}$$

$$I_\nu = \delta^3 [h\nu' \frac{dN'}{dt'\, d\nu' d\Omega' dA'}] \ = \ \delta^3 \ I'_{\nu'}$$

$$I = \int I_\nu d\nu = \int \delta^3 I'_{\nu'} d\nu = \int \delta^3 I'_{\nu'} (\frac{d\nu}{d\nu'}) d\nu'$$

$$I = \delta^3 \int I'_{\nu'} \delta d\nu' = \delta^4 \int I'_{\nu'} d\nu' = \delta^4 I'$$

- **Emissivity**

The (frequency integrated) emissivity j is the

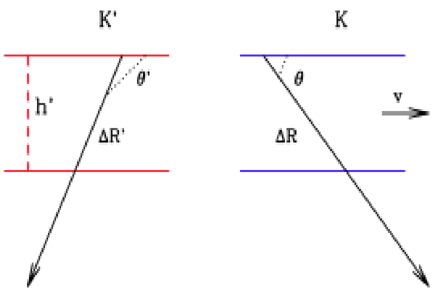

Figure 7

energy emitted per unit time, solid angle and volume. We generally have that the intensity, for an optically thin source, is $I = \int_{\Delta R} j dr$, where ΔR is the length of the region containing the emitting particles.
The emissivity ; $j = h\nu \frac{dN}{dV\,dt\,d\Omega}$. The spatial volume: $dV = dAdL = \delta dL' dA' = \delta dA' dL' = \delta dV'$. Then :

$$j = h\nu \frac{dN}{dV\,dt\,d\Omega} = h(\delta\nu') \frac{dN'}{(\delta dV')(\frac{dt'}{\delta})(\frac{d\Omega'}{\delta^2})}$$

$$j = \delta^3 h\nu' \frac{dN'}{dV' dt' d\Omega'} = \delta^3 j'$$

To understand the transformation properties of emissivity consider a slab with plasma moving with a velocity parallel to the walls (figure 7).
The observer in K will measure a ΔR which depends on his viewing angle. In K' the same path has a different length, because of the aberration of light. The height of the slab $h' = h$, since it is perpendicular to the velocity. In K, the light ray travels a distance : $\Delta R = \frac{h}{\sin\theta}$ and in K' we have : $\Delta R' = \frac{h'}{\sin\theta'}$. $\Delta R' = \frac{1}{\delta}\Delta R$. Therefore the column of plasma contributing to the emission, for $\delta > 1$, is less than what the observer in K would guess by measuring ΔR.
For simplicity, assume a homogeneous plasma :

$$I = j\Delta R = (\delta^3 j')(\delta \Delta R') = \delta^4 j' \Delta R' = \delta^4 I'$$

$$\boxed{I = j\Delta R = \delta^4 j' \Delta R'} \quad \boxed{j = \delta^4 (\frac{\Delta R'}{\Delta R}) = \delta^4 (\frac{1}{\delta}) = \delta^3 j'}$$

And the corresponding transformation for the specific emissivity is $j(\nu) = \delta^2 j'(\nu')$.

◦ **Example: Blobby jet from AGN**

Consider that within a distance R from the apex of a jet (R measured in K), at any given time there are N blobs, moving with a velocity $v = \beta c$ along the jet. To fix the ideas, let assume that beyond R they switch off. If the viewing angle is $\theta = 90^o$, the photons emitted by each blob travel the same distance to reach the observer, who will see all the 10 blobs. But if $\theta < 90^o$, the photons produced by the rear blobs must travel for a longer distance in order to reach the observer, and therefore they have to be emitted before the photons produced by the front blob. The observer will then see less blobs. To be more quantitative, consider a viewing angle $\theta < 90^o$. Photons emitted by blob number 3 to reach blobs number 1 when it produces its last photon (before to switch off) were emitted when the blobs itself was just born (it was crossing point A). They travelled a distance $R\cos\theta$ in a time Δt. During the same time, the blob number 3 travelled a distance $\Delta R = \beta\, c\, \Delta t$ in the forward direction. The fraction f of the blobs that can be seen is :

$$f = \frac{R - \Delta R}{R} = 1 - \frac{\Delta R}{R} = 1 - \frac{\beta c \Delta t}{R} = 1 - \frac{\beta R \cos\theta}{R}$$

$$\boxed{f = 1 - \beta\cos\theta = \frac{\Gamma(1 - \beta\cos\theta)}{\Gamma} = \frac{1}{\Gamma}\frac{1}{\delta}}$$

The bottom line is the following: even if the flux from a single blob is boosted by δ^4, if the jet is made by many (N) equal blobs, the total flux is not just boosted by $N\delta^4$ times the intrinsic flux of a blob, because the observer will see less blobs if $\theta < 90^o$.

- **Flux from moving sources**

The relativistic Doppler Factor is defined by :

$$\delta = \frac{1}{\Gamma\,(1 - \beta\,\cos\theta)} \quad Thus: \; I_\nu = [\frac{1}{\Gamma(1 - \beta\,\cos\theta)}]^3 I'_{\nu'}$$

From the lecturer notes we have : $\quad I'_{\nu'} \propto \nu'^{-\alpha}$

$$I'_{\nu'} \propto \nu^{-\alpha} \Rightarrow I_\nu \propto [\frac{1}{\Gamma\,(1 - \beta\,\cos\theta)}]^3 \nu'^{-\alpha}$$

$$I_\nu \propto [\frac{1}{\Gamma(1 - \beta\,\cos\theta)}]^3 (\frac{\nu}{\delta})^{-\alpha} = [\frac{1}{\Gamma(1 - \beta\,\cos\theta)}]^3 \delta^\alpha \nu^{-\alpha}$$

$$I_\nu \propto [\frac{1}{\Gamma\,(1 - \beta\,\cos\theta)}]^3 [\frac{1}{\Gamma(1 - \beta\,\cos\theta)}]^\alpha \,\nu^{-\alpha}$$

$$\boxed{I_\nu \propto [\frac{1}{\Gamma\,(1 - \beta\,\cos\theta)}]^{3+\alpha} \nu^{-\alpha}}$$

• If the jet is moving towards the observer: $\theta = 0°$ $\cos\theta = 1$. Then :

$$\boxed{I_\nu(\theta = 0°) \propto [\frac{1}{\Gamma(1 - \beta)}]^{3+\alpha} \nu^{-\alpha}}$$

• If the jet is moving away from the observer (receding jet): $\theta = 180° \; \cos\theta = -1$. Then :

$$\boxed{I_\nu(\theta = 180°) \propto [\frac{1}{\Gamma(1+\beta)}]^{3+\alpha}\nu^{-\alpha}}$$

$$\frac{I_\nu(\theta = 0°)}{I_\nu(\theta = 180°)} \propto [\frac{1+\beta}{1-\beta}]^{3+\alpha}$$

$$\frac{I_\nu(\theta = 0°)}{I_\nu(\theta = 180°)} \gg 1 \quad \beta \to 1$$

The emission from the part of the jet measuring towards observer ($\theta \to 0^o$) is **highly Doppler boosted** and therefore much brighter than the part of the jet moving away from the observer.

Let's use a more general example: when the approaching and receding velocity is not directly towards ($\theta = 0^o$) and away ($\theta = 180^o$).
The intensity for the part of the *jet towards* observer:

$$I_{1\nu}(towards) \propto [\frac{1}{\Gamma\,(1-\beta\,\cos\theta)}]^{3+\alpha}\nu^{-\alpha}$$

For a *source receding* the angle will be $\theta_r = \pi + \theta$.

$$I_{2\nu}(away) \propto [\frac{1}{\Gamma\,(1-\beta\cos\theta_r)}]^{3+\alpha}\nu^{-\alpha}$$

$$I_{2\nu}(away) \propto [\frac{1}{\Gamma\,(1+\beta\cos\theta)}]^{3+\alpha}\nu^{-\alpha}$$

$$\boxed{\frac{I_{1\nu}(towards)}{I_{2\nu}(away)} \propto [\frac{1+\beta\cos\theta}{1-\beta\cos\theta}]^{3+\alpha}}$$

For $\beta \to 1$, we have : $\sin\theta = \frac{1}{\Gamma} \Rightarrow \sin^2\theta = \frac{1}{\Gamma^2}$
$1-\cos^2\theta = \frac{1}{\Gamma^2} \Rightarrow \cos^2\theta = 1-\frac{1}{\Gamma^2} = 1-(1-\beta^2) = \beta^2$
Then $\cos\theta = \beta$. In this limit :

$$\frac{I_{1\nu}(towards)}{I_{2\nu}(away)} \propto [\frac{1+\beta\cos\theta}{1-\beta\cos\theta}]^{3+\alpha} \propto (\frac{1+\beta^2}{1-\beta^2})^{3+\alpha}$$

$$\boxed{\frac{I_{1\nu}(towards)}{I_{2\nu}(away)} \propto (\Gamma^2(1+\beta^2))^{3+\alpha} = (2\Gamma^2)^{3+\alpha} \quad \beta \to 1}$$

The approaching component is much brighter than the receding component. This is why *most of the jets* from **AGN** appear to be **one-sided.**

• Moving in an homogeneous radiat^o field

From the reference frame K' the Doppler factor is:

$$\delta' = \frac{1}{\Gamma(1-\beta\cos\theta')}$$

The intensity coming from each element is seen boosted as:

$$I' = \delta'^4 I$$

The radiation energy density is :

$$U' = \frac{1}{c}\int I' d\Omega' = \frac{2\pi}{c}\int_\beta^1 I' d\cos\theta' = \frac{2\pi}{c}\int_\beta^1 \delta'^4 I d\cos\theta'$$

$$U' = \frac{2\pi\, I}{c\Gamma^4}[\frac{1}{3\beta}(1-\beta\cos\theta')^{-3}]_{\beta^1}$$

$$\boxed{U' = (1+\beta+\frac{\beta^2}{3})\Gamma^2\frac{2\pi\, I}{c} = (1+\beta+\frac{\beta^2}{3})\Gamma^2\, U}$$

Do tutorial 4 & 5

⊙ - Chap 4 : Synchrotron emiss^o & absorpt^o - ⊙

• Emission from many electrons

The most probable particle energy distribution in high energy astrophysics is *the power-law*:

$$N(\gamma) = K\gamma^{-P} \qquad [m^{-3}\,\gamma^{-1}]$$

$$N(\gamma) = \frac{dN}{d\gamma} \Rightarrow N(\gamma)d\gamma = \frac{dN}{d\gamma}d\gamma = K\gamma^{-P}d\gamma$$

We assume the pitch angle $\theta \sim 0^o$. The differential emissivity within a frequency interval $d\nu$ is:

$$\epsilon_s(\nu)d\nu = \frac{1}{4\pi}P_s N(\gamma)d\gamma$$

But : $\nu_s = \gamma^2\nu_L \Rightarrow \gamma = (\frac{\nu}{\nu_L})^{1/2} \Rightarrow \frac{d\gamma}{d\nu} = \frac{1}{2}(\frac{\nu^{-1/2}}{\nu_L^{1/2}})$
With $\nu_L = \frac{eB}{2\pi m_e c}$.

$$\epsilon_s(\nu) = \frac{1}{4\pi}P_s N(\gamma)\frac{d\gamma}{d\nu} \propto \frac{1}{4\pi}(\gamma^2 B^2)K\gamma^{-P}\frac{1}{2}(\frac{\nu^{-1/2}}{\nu_L^{1/2}})$$

$$\epsilon_s(\nu) \propto K\,(\frac{\nu}{\nu_L})B^2(\frac{\nu}{\nu_L})^{-\frac{P}{2}}(\frac{\nu}{\nu_L})^{-1/2}\,\nu_L^{-1/2}\nu_L^{-1/2}$$

$$\epsilon_s(\nu) \propto K\,\nu_L^{-1}B^2(\frac{\nu}{\nu_L})^{-(\frac{P-1}{2})} \propto K\,B^2\nu_L^{(\frac{P-3}{2})}\nu^{-(\frac{P-1}{2})}$$

Since $\nu_L = \frac{eB}{2\pi m_e c} \;\Rightarrow\; \nu_L \propto B$.

$$\boxed{\epsilon_s(\nu) \propto K\,B^{(\frac{P-3}{2})}B^2\nu^{-(\frac{P-1}{2})} \propto K\,B^{(\frac{P+1}{2})}\nu^{-(\frac{P-1}{2})}}$$

The *synchrotron flux* measured from a homogeneous and thin source with volume $V \propto R^3$ at a distance d_L is determined as follows ($\alpha = \frac{P-1}{2}$ and $\theta_s = \frac{R}{d_L}$):

$$F_s(\nu) \approx \frac{L_s(\nu)}{4\pi d_L^2} \approx \frac{4\pi\epsilon_s(\nu)V}{4\pi d_L^2} \approx 4\pi\epsilon_s(\nu) \times \frac{V}{4\pi d_L^2}$$

$$F_s(\nu) \propto K\,B^{(\frac{P+1}{2})}\nu^{-(\frac{P-1}{2})} \times \frac{R^3}{d_L^2} \propto K\,\frac{R^2}{d_L^2}R\,\nu^{-\alpha}B^{1+\alpha}$$

$$\boxed{F_s(\nu) \propto \kappa\,\theta_s^2\,R\,B^{1+\alpha}\nu^{-\alpha}}$$

• **Synchrotron absorption: photons**

All emission processes have their own absorption counterpart. For synchrotron radiation the emission is done by relativistic electrons and they don't have a pure Maxwellian distribution. We could then have expected the same behaviour as for an optically thick black body : $I_\nu \propto \nu^2 T$.
But for relativistic electrons inside the plasma we expect an equilibrium between the energy of the thermal particles ($\sim kT$) :

$$kT \sim \gamma m_e c^2 \sim m_e c^2 (\frac{\nu}{\nu_L})^{1/2}$$

We showed earlier that predominantly photons for which $h\nu \ll kT$ are absorbed. In this limit:

$$I_\nu = 2kT\frac{v^2}{c^2} = 2[\gamma m_e c^2]\frac{\nu^2}{c^2} = 2[(\frac{\nu}{\nu_L})^{1/2} m_e c^2]\frac{\nu^2}{c^2}$$

$$I_\nu = \frac{2m_e c^2}{c^2}\nu_L^{-1/2}\nu^{5/2} \qquad \nu_L \propto B \qquad \boxed{I_\nu \propto B^{-1/2}\nu^{5/2}}$$

• **From thick to thin : Turnover Frequency**

To describe the transition from self-absorbed ($\tau_\nu \gg 1$) to optically thin ($\tau_\nu \ll 1$), we make use of the radiative transfer equation:

$$I_\nu = \frac{j_\nu}{\alpha_\nu}(1-e^{-\tau_\nu}) = \frac{\epsilon(\nu)}{\kappa_\nu}(1-e^{-\tau_\nu}) = \frac{\epsilon(\nu)R}{\tau_\nu}(1-e^{-\tau_\nu})$$

$\epsilon(\nu) = j_\nu,\ \kappa_\nu = \alpha_\nu,\ \tau_\nu = R\,\kappa_\nu \Rightarrow \quad \kappa_\nu = \frac{\tau_\nu}{R},$

○ For $\tau_\nu \gg 1$ (self absorbed regime), we simply have :

$$I_\nu = \frac{\epsilon(\nu)R}{\tau_\nu} = \frac{\epsilon(\nu)}{\kappa_\nu}$$

The absorption coefficient, (in cm^{-1}), is for $\tau_\nu \gg 1$:

$$\boxed{\kappa_\nu = \frac{\epsilon(\nu)}{I_\nu} \propto \frac{K\ B^{(\frac{P+1}{2})}\nu^{-(\frac{P-1}{2})}}{B^{-1/2}\nu^{5/2}} = K\ B^{(\frac{P+1}{2})}\ \nu^{-(\frac{P+4}{2})}}$$

The absorption coefficient is depending strongly on frequency. One can see that, as expected, at large frequency the absorption is negligible.
The transition between optically thick ($\tau_\nu \gg 1$) and optically thin ($\tau_\nu \ll 1$) occurs if $\tau_\nu = 1$. This will define the so-called **turnover frequency**, i.e:

$$\tau_{\nu t} = R\kappa_{\nu t} = 1 \qquad R\ K\ B^{(\frac{P+1}{2})}\ \nu_t^{-(\frac{P+4}{2})} = 1$$

$$\nu_t^{-(\frac{P+4}{2})} \propto \frac{B^{(\frac{P+1}{2})}}{R\ K} \qquad \boxed{\nu_t \propto [R\ K\ B^{(\frac{P+1}{2})}]^{2/(P+4)}}$$

⊙ - Chap 5 : Compton scattering - ⊙

• **The Klein-Nishina cross section**

The Thomson cross section is the classical limit of the more general Klein Nishina cross section.

$$\sigma_{KN} = \frac{3}{4}\sigma_T\{\frac{1+x}{x^3}\ [\frac{2x(1+x)}{1+2x} - \ln(1+2x)]\}$$

$$+\frac{3}{4}\sigma_T\{\frac{\ln(1+2x)}{2x} - \frac{1+3x}{(1+2x)^2}\}$$

Approximations for the non-relativistic and the ultra-relativistic limit are:

○ For $x \ll 1$: $\sigma_{KN} \simeq \sigma_T(1 - 2x + \frac{26x^2}{5} + \cdots)$

○ For $x \gg 1$: $\sigma_{KN} \simeq \frac{3}{8}\frac{\sigma_T}{x}\ [\ln(2x) + \frac{1}{2}]$

With $\sigma_T = \frac{8\pi}{3}r_0^2$ the Thomson cross section. ($r_0 = \frac{e^2}{m_e\,c^2} = 2.82 \times 10^{-13}\ cm$ is the classic electron radius.)

Compton scattering is more effective in Thomson

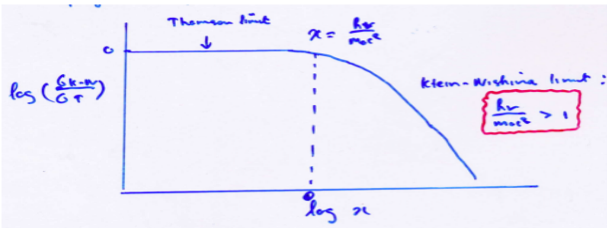

Figure 8

limit since the cross-section has a maximum value, i.e σ_T. Scattering between electrons and photons with $x > 1$ is less effective since $\sigma_{KN} \to 0$ in that limit.

• **Thomson & $K - N$ limits**

We transform to a reference frame where the electron is in rest (frame co-moving with e^-) and evaluate the incoming photon energy in that frame. If:

$$x' = \frac{h\nu'}{m_e c^2} < 1 \quad \Rightarrow \quad Thomson\ \ limit$$

$$x' = \frac{h\nu'}{m_e c^2} > 1 \quad \Rightarrow \quad K - N\ \ limit$$

The energy of the incoming photon in the rest frame of the electron will be Doppler shifted. If $\cos\theta' \to \beta$:

$$[\Gamma = \gamma];\ \nu' = \frac{\nu}{\Gamma(1-\beta\cos\theta')} = \frac{\nu}{\Gamma(1-\beta^2)} = \Gamma\nu = \gamma\nu$$

Therefore in the frame co-moving with electron the incoming photons (within θ') will all be Doppler boosted. Therefore for Inverse-Compton scattering we have (using the fact that $\nu' = \gamma\nu$) :

$$Thomson\ \ limit\ :\qquad \nu < \frac{m_e c^2}{\gamma h}$$

$$K - N\ \ limit\ :\qquad \nu > \frac{m_e c^2}{\gamma h}$$

• Scattering Energy : Thomson Regime

In K' the e^- sees :

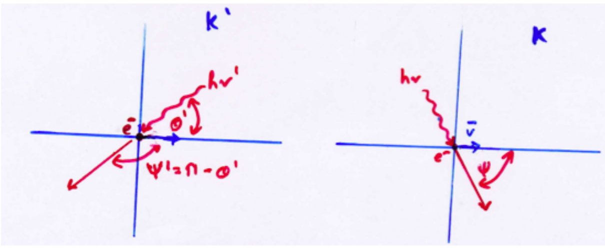

Figure 9

$$\nu' = \frac{\nu_{em}}{\Gamma(1-\beta\cos\theta')} = \frac{\nu_{em}}{\Gamma(1+\beta\cos(\pi-\theta'))} = \frac{\nu_{em}}{\Gamma(1+\beta\cos\Psi')}$$

$$\nu' = \frac{\nu_{em}}{\Gamma(1+\beta[\frac{\cos\Psi-\beta}{1-\beta\cos\Psi}])} = \frac{(1-\beta\cos\Psi)\nu_{em}}{\Gamma(1-\beta^2)}$$

,

$$\boxed{\nu' = \Gamma(1-\beta\cos\Psi)\nu_{em}}$$

A very convenient way to treat Inverse-Compton scattering is to evaluate the process in the co-moving reference frame. When we take : $x' = \frac{h\nu'}{m_e c^2} < 1$, we are in the **Thomason limit** and then we know the scattering process is *elastic* and $\Rightarrow$ $\boxed{x'_1 = x'}$ regardless of angle.

Note that : $x'_1 =$ *scattered photon energy in* K'.

. $x' =$ *incoming photon energy in* K'.

The photon will be scattered at on angle Ψ'_1 with respect to the electron velocity. One can then use the Doppler equation to transform this scattered energy (x'_1) and angle (Ψ'_1) to the lab frame (K). This is done in the following way:

$$x_1 = \frac{h\nu_1}{m_e c^2} = \delta(\frac{h\nu'}{m_e c^2}) = \delta x'_1\ ;\ \ \delta = \frac{1}{\Gamma(1-\beta\cos\Psi_1)}$$

$$x_1 = \delta x'_1 = \frac{1}{\Gamma(1-\beta\cos\Psi_1)}x'_1 = \frac{1}{\Gamma(1-\beta[\frac{\cos\Psi_1+\beta}{1+\beta\cos\Psi_1}])}x'_1$$

$$x_1 = \Gamma(1+\beta\cos\Psi'_1)x'_1$$

We can also express all quantities in that form:

$$\boxed{x'_1 = x' = \Gamma(1-\beta\cos\Psi)x}$$

This gives: $x_1 = \delta x'_1 = \frac{1}{\Gamma(1-\beta\cos\Psi_1)}x'$

$$\boxed{x_1 = \frac{1}{\Gamma(1-\beta\cos\Psi_1)} \times \Gamma(1-\beta\cos\Psi)x = (\frac{1-\beta\cos\Psi}{1-\beta\cos\Psi_1})x}$$

• Maximum Energy Transfer

The maximum energy transfer from the e^- to the scattered photon is if the collision is **head-on** and the photon is scattered in opposite direction is from where it came.

Incoming photon angle relative to e^- : $\Psi = \pi$

Scattered photon angle relative to e^- : $\Psi_1 = 0$

$(x_1)_{max} = (\frac{1-\beta\cos\pi}{1-\beta\cos 0})x = (\frac{1+\beta}{1-\beta})x = \Gamma^2(1+\beta)^2\ x$

For $\beta \to 1$ $\boxed{(x_1)_{max} = 4\gamma^2\ x}$

• Minimum Energy Transfer

The minimum energy transfer from e^- to photon is during a *head-tail* collision. Here the photon approaches the e^- from behind and scattered in opposite directions:

Incoming photon angle relative to e^- : $\Psi = 0$

Incoming photon angle relative to e^- : $\Psi_1 = \pi$

$(x_1)_{max} = (\frac{1-\beta\cos 0}{1-\beta\cos\pi})x = (\frac{1-\beta}{1+\beta})x = \frac{1}{\Gamma^2(1+\beta)^2}\ x$

For $\beta \to 1$ $\boxed{(x_1)_{max} = \frac{1}{4\gamma^2}\ x}$

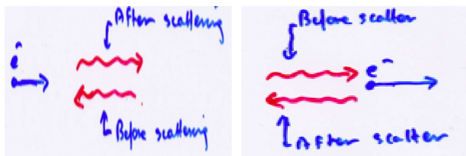

Figure 10: *Maximum and Minimum Energy Transfer*

• Isotropic photon distribution

We showed earlier that a co-moving observer in K' see on isotropic energy density of the background photons $U' = \frac{4}{3}\Gamma^2 U_{iso}$ $\beta \to 1$

If the observer in K' is co-moving with an e^- moving with high velocity ($\beta \to 1$) thus :

$U' = \frac{N'<x'>}{V'}$; $U_{iso} = \frac{N<x>}{V}$; $\frac{N'<x'>}{V'} = \frac{4}{3}\Gamma^2\frac{N<x>}{V}$

We know: $V' = A'l' = A\Gamma l = \Gamma Al = \Gamma V$; $\frac{V'}{V} = \Gamma$

In the co-moving frame the e^- scatters the photon elastically in the **Thomson limit**:

$< x' > = < x'_1 > \Rightarrow \frac{<x'_1>}{<x>} = \frac{4}{3}\Gamma^2 \times \Gamma;$ $\boxed{\frac{< x'_1 >}{< x >} = \frac{4}{3}\Gamma^3}$

But the relation between the scattered energy in K' and K is : $< x'_1 > = \Gamma(1-\beta < \cos\Psi >) < x_1 >$

Since $< \cos\Psi >= 0$; $\quad < x_1' >= \Gamma < x_1 >$
$\frac{<x_1'>}{<x>} = \Gamma \frac{<x_1>}{<x>} = \frac{4}{3}\Gamma^3 \quad \Rightarrow \quad \frac{<x_1>}{<x>} = \frac{4}{3}\Gamma^2$

$$\boxed{< x_1 >= \frac{4}{3}\Gamma^2 < x >}$$

We can simply calculate the rate of scatterings per electron considering all quantities in the lab frame. Let $n(\epsilon)$ be the density of photons of energy $\epsilon = h\nu$, v the electron velocity and Ψ the angle between the electron velocity and the incoming photon. For monodirectional photon distributions, we have:

$$\frac{dN}{dt} = \int \sigma_T v_{rel} n(\epsilon) d\epsilon = \int \sigma_T c (1 - \frac{v}{c}\cos\Psi) n(\epsilon) d\epsilon$$

$$\frac{dN}{dt} = \int \sigma_T c (1 - \beta\cos\Psi) n(\epsilon) d\epsilon; \; v_{rel} = c - v\cos\Psi$$

The power contained in the scattered radiation is :

$$\frac{d\epsilon_\gamma}{dt} = \epsilon_1 \frac{dN}{dt}$$

ϵ_1 : energy of scattered photon, and $\frac{dN}{dt}$: the scattered rate. Let $\quad x_1 = \frac{\epsilon_1}{m_e c^2}, \quad x = \frac{\epsilon}{m_e c^2}$
$x_1 = \frac{\epsilon_1}{m_e c^2} = x(\frac{1-\beta\cos\Psi}{1-\beta\cos\Psi_1}) = \frac{\epsilon}{m_e c^2}(\frac{1-\beta\cos\Psi}{1-\beta\cos\Psi_1})$

$$\Rightarrow \quad \epsilon_1 = \epsilon(\frac{1-\beta\cos\Psi}{1-\beta\cos\Psi_1})$$

$$\frac{d\epsilon_\gamma}{dt} = \epsilon_1 \frac{dN}{dt} = \int \epsilon_1 c\,\sigma_T (1 - \beta\cos\Phi) n(\epsilon) d\epsilon$$

$$\frac{d\epsilon_\gamma}{dt} = c\,\sigma_T \int \epsilon \frac{(1-\beta\cos\Psi)^2}{1-\beta\cos\Psi_1} n(\epsilon) d\epsilon$$

$$\frac{d\epsilon_\gamma}{dt} = c\,\sigma_T\,\gamma^2 \int (1-\beta\cos\Psi)^2 \epsilon\, n(\epsilon) d\epsilon$$

$$< \frac{d\epsilon_\gamma}{dt} >= c\,\sigma_T\,\gamma^2 \int < (1-\beta\cos\Psi)^2 > \epsilon\, n(\epsilon) d\epsilon$$

$< \cos\Psi >= 0; \quad < \cos^2\Psi >= \frac{1}{3}$

$$< \frac{d\epsilon_\gamma}{dt} >= c\,\sigma_T\,\gamma^2 \int (1 + \frac{\beta^2}{3}) \epsilon\, n(\epsilon) d\epsilon$$

$$< \frac{d\epsilon_\gamma}{dt} >= (1 + \frac{\beta^2}{3})\, c\,\sigma_T\,\gamma^2 U_{ph}; \;\; U_{ph} = \int \epsilon\, n(\epsilon) d\epsilon$$

$$< \frac{d\epsilon_\gamma}{dt} >= (1 + \frac{\beta^2}{3})\, c\,\sigma_T\,\gamma^2\,\epsilon_{ph} n_{ph}$$

$$< \frac{d\epsilon_\gamma}{dt} >= [(1 + \frac{\beta^2}{3})\gamma^2\,\epsilon_{ph}] \times [c\,\sigma_T\,n_{ph}]$$

This is the power contained in scattered radiation. This is added on top of the already existing background photon field. **Therefore to probe "only" the Inverse-Compton power we subtract the background field** $\sigma_T\,c\,n_{ph}$.

$P_c(\gamma) = \frac{d\epsilon_\gamma}{dt} - \sigma_T\,c\,n_{ph} = (1 + \frac{\beta^2}{3})\,c\,\sigma_T \gamma^2 U_{ph} - \sigma_T c\,n_{ph}$
$P_c(\gamma) = \frac{(1+\frac{1}{3})\beta^2\,c\,\sigma\,U_{ph}}{1-\beta^2} = \frac{4}{3}\gamma^2\beta^2\,c\,\sigma_T\,U_{ph}$

$$\beta \;\to\; 1 \quad \boxed{P_c(\gamma) = [\frac{4}{3}\gamma^2\,\epsilon_{ph}] \times [c\,\sigma_T\,n_{ph}] =< \epsilon_1 >< rate >}$$

$* \; P_c(\gamma) = \frac{4}{3}\sigma_T\,c\,\gamma^2\beta\,U_{ph}$; $P_{syn}(\gamma) = \frac{4}{3}\sigma_T c\,\gamma^2\beta U_{mag}$
The γ-ray production rate :

$$\frac{d\epsilon_\gamma}{dt} = [c\,\sigma\,\gamma^2 \int \epsilon n(\epsilon) d\epsilon]\;(1 - \beta\cos\Psi)^2 \;\Rightarrow$$

$\frac{d\epsilon_\gamma(\cos\Psi)}{dt} = const\;(1 - \beta\cos\Psi)^2; \quad \mu = \cos\Psi \;(in\;lab)$

$$\Rightarrow \; P_c(\mu) = const \int (1 - \beta\,\mu)^2 d\mu$$

$$\boxed{\frac{P_{\gamma_{up}}}{P_{\gamma_{down}}} = \frac{2\int_{-1}^{0}(1-\beta\,\mu)^2 d\mu}{2\int_{o}^{1}(1-\beta\,\mu)^2 d\mu} = 7\;; \;\; \beta \to 1}$$

The scattered rate with the up-stream photons is 7−times higher than with the photons coming from the back.

$$\boxed{\frac{P_{\gamma_{up}}}{P_{\gamma_{side}}} = \frac{2\int_{-1}^{0}(1-\beta\,\mu)^2 d\mu}{2\int_{-1}^{1}(1-\beta\,\mu)^2 d\mu} = \frac{7}{4}\;; \;\; \beta \to 1}$$

- **Cooling time and Compactness**

The cooling time due to the inverse Compton process is :

$$t_{IC} = \frac{E}{P_c(\gamma)} = \frac{\gamma m_e c^2}{\frac{4}{3}\sigma_T\,c\,\gamma^2\,\beta^2\,U_{ph}} = \frac{3 m_e c^2}{4\sigma_T\,c\,\gamma\,U_{ph}}; \; \beta \to 1$$

This equation offers the opportunity to introduce an important quantity, namely the **compactness** of an astrophysical source, that is essentially the luminosity L over the size R ratio. Consider in fact how U_{ph} and L are related: $\qquad U_{ph} = \frac{L}{4\pi R^2\,c}$.

Although this relation is almost universally used, there are subtleties. It is surely valid if we measured U_{ph} *outside* the source, at a distance R from its center. In this case $4\pi R^2 c$ is simply the volume of the shell crossed by the source radiation in one second. But if we are *inside* an homogeneous, spherical transparent source, a better way to calculate U_{ph} is to think to the average time needed to the typical photon to exit the source. This is $t_{esc} = 3R/(4c)$. It is less than R/c because the typical photon is not born at the center (there is more volume close to the surface). If $V = (4\pi/3)R^3$ is the volume, we can write:

$$\boxed{t_{IC} = \frac{3\pi m_e c^2 R^2}{\sigma_T\,\gamma\,L} \to \frac{t_{IC}}{R/c} = \frac{3\pi}{\gamma}\frac{m_e c^3 R}{\sigma_T\,L} = \frac{3\pi}{\gamma}\frac{1}{l}}$$

Where the *compactness parameter* l is : $\boxed{l = \frac{\sigma_T L}{m_e c^3 R}}$

For $l > 1$, $\frac{t_{IC}}{R/c} < 1 \Rightarrow$ we have that even low energy electrons cool by the Inverse Compton process in less than a light crossing time R/c.

• Single particle spectrum

We will give the main ideas behind the fact that the energy (frequency) of the scattered photons are $\sim \gamma^2$ times that of the background photons. There are a few steps to consider:

∘ Assume that the relativistic electron travels in a region where there is a radiation energy density Ur made by photons which we will take, for simplicity, monochromatic, therefore all having a *dimensionless frequency* $x = h\nu/m_e c^2$.

∘ In the frame where the electron is at rest, half of the photons appear to come from the front, within an angle $\theta' = 1/\gamma$.

∘ The typical frequency of these photons is $x' \sim \gamma x$ (it is twice that for photons coming exactly head on).

∘ If we are in Thomson regime (elastic scatter in e^- rest frame) then: $x' = \frac{h\nu'}{m_e c^2} < 1 = x_1'$.

∘ All the photons in K' that were scattered within $\Psi' <= \frac{\pi}{2}$, now have angles viewed from the lab system $\Psi_1' <= \frac{1}{\gamma}$

∘ The lab energy of this scattered photon is then obtained from the Doppler equation: $x_1 = \delta x_1'$

$x_1 = \frac{x_1'}{\gamma[1-\beta\cos\Psi_1]} = \frac{x_1'}{\gamma[1-\beta^2]} = \gamma x_1'$

$< x_1 >= \gamma < x_1' >= \gamma[< x' >] = \gamma^2 x$

A detailed analysis gives the IC emissivity:

$$\epsilon_{IC}(x_1) = \frac{\sigma_T n\, I_0\,(1+\beta 0}{4\gamma^2\beta^2 x_0}\; F_{IC}(x_1)$$

$$F_{IC} = \frac{x_1}{x_0}[\frac{x_1}{x_0} - \frac{1}{(1+\beta)^2\gamma^2}];\;\; \frac{1}{(1+\beta)^2\gamma^2} < \frac{x_1}{x_0} < 1$$

$F_{IC} = \frac{x_1}{x_0}[1 - \frac{x_1}{x_0}\frac{1}{(1+\beta)^2\gamma^2}];\; 1 < \frac{x_1}{x_0} < (1+\beta)^2\gamma^2$

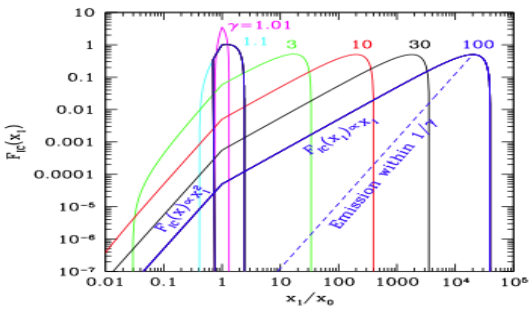

Figure 11: *$x_1/x_0 < 1$: scattered photons have less energy than incoming ones. $x_1/x > 1$: Scattered photons have more energy than incoming ones.*

⊙ - Chap 6 : Synchrotron SelfCompton - ⊙

• SSC emissivity

The importance of this process will obviously be high for system with *i)* high electron density (relativistic), *ii)* photon density and *iii)* high magnetic field energy density. If the relativistic electron distribution is a power-law:

$$N(\gamma) = K\gamma^{-p}$$

We expect the SSC flux to scale as K^2 (e^-s work twice), i.e quadratic in the electron density.
If the photo field is produced by synchrotron emitting we should subtract the appropriate expression for the synchrotron radiation energy density:

$$U_{syn} = \frac{L_\nu(ergs^{-1}Hz^{-1})}{V} \times t_{sec} = \frac{3R}{4c}\frac{L_\nu}{V}$$

$$\boxed{U_{sy}(\nu) = 4\pi\frac{3R}{4c}[\frac{1}{4\pi}\frac{L_\nu}{V}] = 4\pi\,\frac{3R}{4c}\,\epsilon_{syn}(\nu)}$$

$$\epsilon_{IC}(\nu_{IC}) = \frac{1}{4\pi}\frac{(\frac{4}{3})^\alpha}{2}\frac{\tau_c}{R/c}\nu_c^{-\alpha}\int_{\nu_{min}}^{\nu_{max}}\frac{U_{ph}}{\nu}\nu^\alpha d\alpha$$

$$\epsilon_{IC}(\nu_{IC}) = \frac{(4/3)^{\alpha-1}}{2}\tau_c\epsilon_{syn,0}\nu_c^{-\alpha}\int_{\nu_{min}}^{\nu_{max}}\frac{d\alpha}{\nu}$$

As you can see, $\epsilon_{syn,0}\nu_c^{-\alpha} = \epsilon_{syn}(\nu_c)$ is nothing else than the specific synchrotron emissivity calculated at the (Campton) frequency ν_c. Furthermore, the integral gives a logarithmic term, that we will call $\ln\Lambda$. We finally have:

$$\boxed{\epsilon_{SSC}(\nu_c) = \frac{(4/3)^{\alpha-1}}{2}\tau_c\epsilon_{syn}(\nu_c)\ln\Lambda}$$

In this form the ratio between the synchrotron and the SSC flux is clear, it is $\frac{(4/3)^{\alpha-1}}{2}\tau_c\ln\Lambda \sim \tau_c\ln\Lambda$. It is also clear that since $\tau_c = \sigma_T RK$ and $\epsilon_{syn}(\nu_c) = KB^{1+\alpha}$, then, as we have guessed, the SSC emissivity $\epsilon_{SSC}(\nu_c) \propto K^2$ (i.e. electrons work twice). Fig. 12 summarizes the main results. .

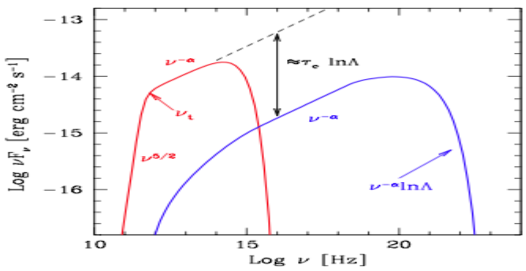

Figure 12: *Typical example of **SSC spectrum**, shown in the νF_ν vs ν representation. The spectral indices instead correspond to the $F_\nu \propto \nu^{-\alpha}$ convention.*