Highly conducting Fluids

- For particle acceleration to occur in astrophysical plasmas the plasma should be collisionless otherwise particle-particle collisions will result in thermalization and not acceleration

- The second requirement is that the plasma should be highly conducting, $\sigma \to \infty$. In this case ideal MHD should be satisfied. In the frame co-moving with the fluid the fields are

Co-moving frame: $[\bar{E}', \bar{B}']$

$$\bar{E}' = \Gamma\left(\bar{E} + \tfrac{1}{c}(\bar{V} \times \bar{B})\right)$$
$$\bar{B}' = \Gamma\left(\bar{B} - \tfrac{1}{c}(\bar{V} \times \bar{E})\right)$$

Stationary frame: $[\bar{E}, \bar{B}]$

Here $\underline{\bar{V} = \bar{\beta} c}$ represents the $\underline{\text{bulk flow velocity of the fluid}}$ for observer in [K]. For $\underline{\underline{\sigma \to \infty}}$ the fluid behaves like a superconductor and $\bar{E}' \to 0$ to move charges, since

$$\bar{F}' = q\bar{E}'$$

can set charges in motion and drive a current. Then for $\bar{E}' \to 0$

$$\Gamma\left(\bar{E} + \tfrac{1}{c}(\bar{V} \times \bar{B})\right) = 0$$

$$\Rightarrow \boxed{\bar{E} = -\tfrac{1}{c}(\bar{V} \times \bar{B})}$$

* Note the $\bar{V}$ represents the velocity of a magnetized plasma with respect to a stationary observer in [K], the Lab. frame.

$$\Gamma \neq 0$$
$$\bar{\beta} = \frac{\bar{V}}{c}$$

Measured from a stationary frame (Lab. frame) the value of the $\bar{E}$-field is given by the Lorentz force $\left(-\tfrac{1}{c}(\bar{V} \times \bar{B}) = -(\bar{\beta} \times \bar{B})\right)$

* $\bar{F}_{m,L} = \frac{q}{c}(\bar{V} \times \bar{B})$ $\Rightarrow$ The $\bar{E}$-field balances the $\frac{1}{c}(\bar{V} \times \bar{B})$ effect.

Now if $\bar{E}' \to 0$, we get

$\bar{E} = -(\bar{\beta} \times \bar{B})$

$$\bar{B}' = \Gamma\left(\bar{B} - \tfrac{1}{c}(\bar{v} \times \bar{E})\right)$$

$$= \Gamma(\bar{B} - (\bar{\beta} \times \bar{E}))$$

$$= \Gamma(\bar{B} - (\underset{\bar{a}}{\bar{\beta}} \times [-(\underset{\bar{b}}{\bar{\beta}} \times \underset{\bar{c}}{\bar{B}})]))$$

$$= \Gamma(\bar{B} + \bar{\beta} \times (\underbrace{\bar{\beta} \times \bar{B}}_{=0 \text{ for } \sigma \to \infty}))$$

$\bar{a} \times (\bar{b} \times \bar{c}) = \bar{b}(\bar{a}\cdot\bar{c}) - \bar{c}(\bar{a}\cdot\bar{b})$

$$= \Gamma(\bar{B} + (\bar{\beta}(\bar{\beta}\cdot\bar{B}) - \bar{B}(\bar{\beta}\cdot\bar{\beta})))$$

$$= \Gamma(\bar{B} - \bar{B}(\bar{\beta}\cdot\bar{\beta}))$$

$$= \Gamma\bar{B}(\underbrace{1 - \beta^2}_{\frac{1}{\Gamma^2}})$$

$$= \frac{1}{\Gamma}\bar{B}$$

$$\Rightarrow \bar{B} = \Gamma\bar{B}'$$

* For an ideal fluid with $\sigma \to \infty$ there is no bulk motion of fluid with respect to $\bar{B}$-field, field is frozen into plasma.

Therefore from

$$\bar{E} = -(\bar{\beta} \times \bar{B})$$

$$= -(\bar{\beta} \times \Gamma\bar{B}')$$

$$= -\Gamma(\bar{\beta} \times \bar{B}')$$

The electric field will play an important role in the acceleration of particles, the magnetic lorentz force dont accelerate particles!! We can show this in the following way.

Magnetic Lorentz Force dont accelerate, only the Electric Field does

The rate of change of KE of a particle as a result of the magnetic component

$$\frac{dE_{KE}}{dt} = \bar{V}_p \cdot \bar{F}_L$$

$$= c\bar{\beta}_p \cdot q \Big[\overbrace{\tfrac{1}{c}(\bar{V}_p \times \bar{B})}^{\text{magnetic component}} \Big]$$

<u>Force on particle:</u>

$$\bar{F}_L = q\bar{E} + \frac{q}{c}(\bar{V}_p \times \bar{B})$$

$$= c\,\bar{\beta}_p \cdot q\,[\,(\bar{\beta}_p \times \bar{B})]$$

$$= cq\;\bar{\beta}_p \cdot (\bar{\beta}_p \times \bar{B})$$

$$\boxed{\text{B-fields cannot increase a particle's KE!!}} = 0 \quad \text{since} \quad \bar{\beta}_p \perp (\bar{\beta}_p \times \bar{B})$$

If the B-fields are transported with the bulk flow ($\bar{V}$) velocity as a result of the Frozen-in condition the B-fields can act as a scattering centre:

$$\frac{dE_{KE}}{dt} = \bar{V}_p \cdot \bar{F}_L$$

$$= \bar{V}_p \cdot q\left[\bar{E} + \left(\frac{\bar{V}_p}{c} \times \bar{B}\right)\right]$$

$$= q\,\bar{V}_p \cdot \bar{E} + \frac{q}{c}\,\bar{V}_p \cdot (\bar{V}_p \times \bar{B})$$

$$= qc\,\bar{\beta}_p \cdot \bar{E} + qc\,\bar{\beta}_p \cdot (\bar{\beta}_p \times \bar{B})$$

$$= qc\,\bar{\beta}_p \cdot \bar{E} + qc\,\bar{\beta}_p \cdot (\bar{\beta}_p \times \bar{B})$$

$$= qc\,\bar{\beta}_p \cdot (-\underset{\uparrow\text{ Fluid velocity}}{\bar{\beta}} \times \bar{B}) + \underbrace{qc\,\bar{\beta}_p \cdot (\bar{\beta}_p \times \bar{B})}_{=0\ \ \bar{\beta}_p \perp (\bar{\beta}_p \times \bar{B})}$$

$$= -qc\,\bar{\beta}_p \cdot (\bar{\beta} \times \bar{B})$$

Using $\bar{A} \cdot (\hat{B} \times \bar{C}) = \hat{B} \cdot (C \times \hat{A}) = \bar{C} \cdot (\hat{A} \times \hat{B})$

Then

$$\frac{d\mathcal{E}_{KE}}{dt} = -qc \, \overset{A}{\bar{\beta}_p} \cdot (\overset{\hat{B}}{\bar{\beta}} \times \overset{\hat{C}}{\bar{B}})$$

$$= -qc \, \bar{\beta} \cdot (\hat{B} \times \hat{\beta}_p)$$

$$= qc \, \bar{\beta} \cdot (\hat{\beta}_p \times \bar{B})$$

$$= \bar{\beta} \cdot [qc(\hat{\beta}_p \times \hat{B})]$$

$$= \bar{\beta} \cdot \bar{F}_{m,L}$$

For $q = -e$ (electron) $\Rightarrow F_{m,L} = -ec(\hat{\beta}_p \times \bar{B})$

$$\Rightarrow \frac{d\mathcal{E}_{KE}}{dt} = \hat{\beta} \, \bar{F}_{m,L} \cos\theta$$

> 0 for $\theta = \pi$ (Head-on collision)

For $q = +e$ (ion) $\Rightarrow F_{m,L} = +ec(\hat{\beta}_p \times \bar{B})$

$$\Rightarrow \frac{d\mathcal{E}_{KE}}{dt} = \beta \, \bar{F}_{m,L} \cos\theta$$

> 0 for $\theta = 0$ (overtaking collision)

So from these simplistic arguments we see that particle acceleration in highly conducting magnetized fluids is in principle a scattering process.

Remember this approximation is valid for most astrophysical environments !!

Hence

$$\frac{d\mathcal{E}_{kc}}{dt} = \bar{\beta} \cdot \bar{F}_{m,c}$$

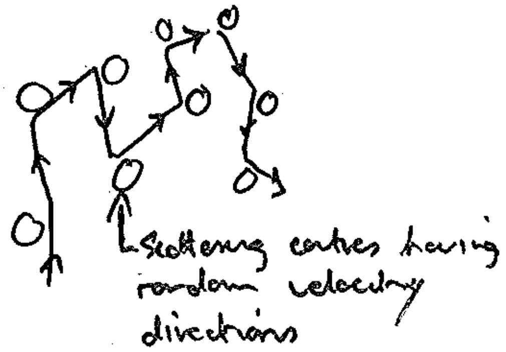
Bulk flow velocity

The importance of bulk flow motion was first realized by Enrico Fermi who developed two simple models for particle acceleration which have served as templates for all subsequent work.

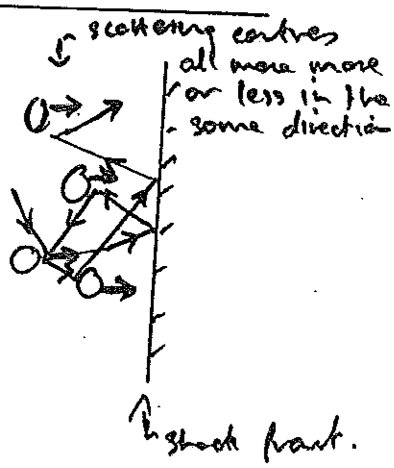
- Stochastic (Fermi II) acceleration by scattering off randomly moving magnetized clouds
- Regular (Fermi I) acceleration by scattering or reflecting off shocks

For stochastic acceleration (Fermi II) the particles diffuse in energy space in such a way that the mean energy increases. This is just because head-on interactions (energy gains) are slightly more than overtaking interactions (energy drains).

Stochastic Fermi II



Regular Fermi I



Maximum energy of cosmic accelerators.

The electromagnetic character of astrophysical particle acceleration in magnetized fluids allows one to place a limit on the energy particle's can reach in cosmic accelerators.
We showed earlier

- $\hat{E} = -\frac{\bar{V}}{c} \times \hat{B} = -\Gamma\left(\frac{v}{c} \times \hat{B}'\right)$

- $\frac{dE_{KE}}{dt} = \hat{V}_p \cdot \bar{F}_L$

magnetic component don't increase KE.

$$= \hat{V}_p \cdot q\bar{E} \longleftarrow E = -\frac{\bar{V}}{c} \times \bar{B}$$

$$\left(\frac{\bar{V}_p}{c} \times \bar{B}\right) \cdot \hat{V}_p \to 0$$

$$= q\,\hat{V}_p \cdot \bar{E}$$

Then

$$\Delta E_{KE} = \int \frac{dE_{KE}}{dt}\,dt$$

$$= \int q\,\hat{V}_p \cdot \hat{E}\,dt$$

$$= q\int \bar{E} \cdot (\hat{V}_p\,dt)$$

$$= q\int \bar{E} \cdot d\bar{s} \qquad d\hat{s} = \hat{V}_p\,dt$$

This then represents the path length that the particle is subjected to $\bar{E}$-field

This becomes

$$\Delta E_{KE} = q\int \bar{E} \cdot d\bar{s}$$

$$= q\int -\left(\frac{\bar{V}}{c} \times \hat{B}\right) \cdot d\bar{s} \qquad \Rightarrow |\hat{\beta} \times \hat{B}| = \beta B \sin\psi$$

$$= \beta B_\perp$$

But note that $\bar{B}$ is just the $\perp$ component in Lab frame

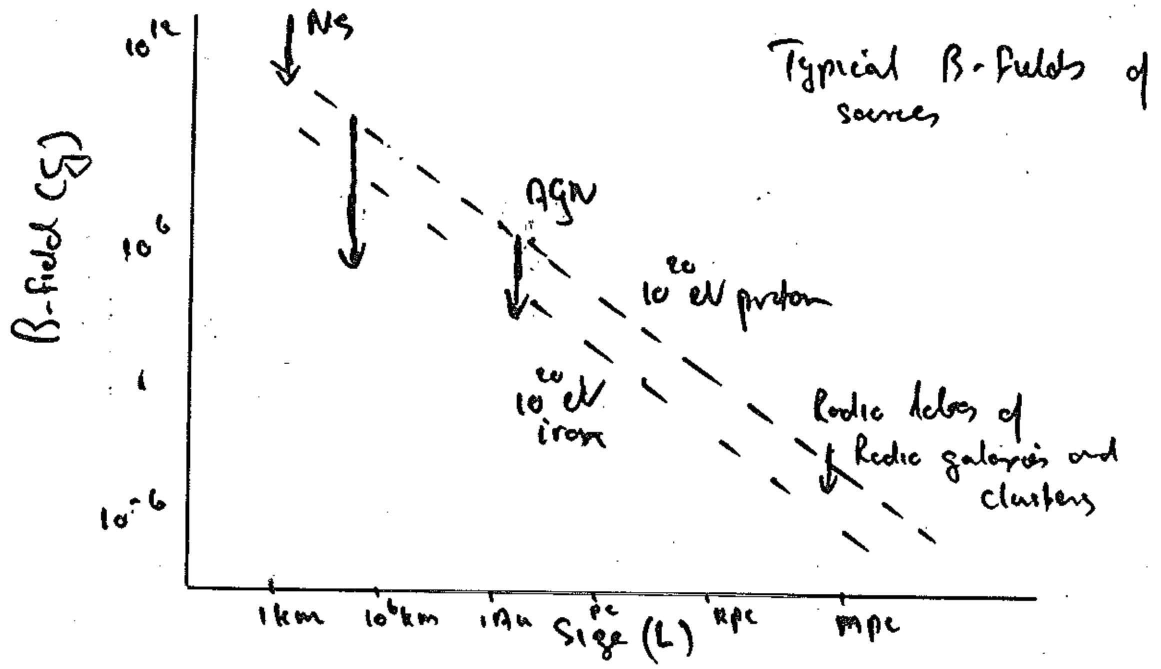
$$\Delta E_{K\varepsilon} = -\frac{q}{c}\int(\bar{V}\times\bar{B}_\perp)\cdot d\bar{s}$$
$$= -\frac{q\Gamma}{c}\int(\bar{V}\times\bar{B}_\perp{}')\cdot d\bar{s}$$

The Lorentz factor Γ represents the Lorentz factor the bulk flow velocity constitutes relative to [K].

To order of magnitude, a source with typical size R_s involving a velocity $V=\beta c$ can produce particles with energy

$$\mathcal{E} \le \mathcal{E}_{max}$$
$$\le q\,\beta_s B_s R_s$$
$$\le q\,\Gamma\,\beta_s\,B'_s R_s \quad \text{magnetic fields in frame of source.}$$
$$\le q\,\Gamma\,\beta_s\,B'_s R_s$$

See Hillas diagram in Achterberg's notes:



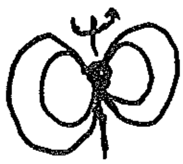
Example: Consider a magnetized neutron star with surface field of $B \rightarrow 10^{12}$ Gauss. The rotation period $P_* \rightarrow 1$ ms. The radius $R_* \rightarrow 10$ km. Determine E-fields generated in magnetosphere

$$E = \frac{1}{c}(\bar{V}_{rot} \times \hat{B})$$

$$V_{rot} \approx \frac{2\pi R_*}{P_*} \approx 6 \times 10^{9} \text{ cm s}^{-1} \sim 0.2c$$

$$E = \bar{\beta}_* \times \bar{B}$$

$$\sim \hat{\beta}_* B$$



$$E = 0.2 \times 10^{12} \text{ Gauss}$$

$$\lessapprox 2 \times 10^{11} \left(\frac{B}{10^{12}\,G}\right) \text{ statvolt} \cdot \text{cm}^{-1}$$

$$\text{Emf} \approx E\,L \times 300 \text{ Volt}$$

$$\lessapprox 10^{19} \left(\frac{E}{2 \times 10^{11}\ \text{s.v cm}^{-1}}\right)\left(\frac{L}{10^{6}\,\text{cm}}\right) \text{ Volt}$$