Synchrotron Emission and Absorption

Many astrophysical sources are magnetized and contain relativistic leptons (electrons and positrons). Magnetic fields and relativistic electrons are the main ingredients for synchrotron radiation. The driving mechanism behind this is the Lorentz force. The Lorentz force makes a particle like an electron gyrate around a magnetic field line.

Exercise: Show that although the Lorentz force does exert work on a particle making it accelerate in a magnetic field, the particle's speed remains constant.

The modus operandi for this section is

1.) Derive an expression for the power emitted by a single electron

2.) Outline the basics of the spectrum emitted by a single e^-

3.) Derive an expression for the spectrum from an ensemble of electrons

4.) Look at synchrotron self absorption.

Total Energy Losses of a single particle

To calculate the total integrated power we consider the problem in a frame where the particle is momentarily in rest. In this frame the velocity is zero but not the acceleration

Making use of the fact that rate of energy loss rate (Power) is Lorentz invariant, we get

$$P_e = P_e' = \frac{2e^2}{3c^3} a'^2 \quad \text{(Larmor's eqn)}$$

$$= \frac{2e^2}{3c^3} \left(a_{\parallel}'^2 + a_{\perp}'^2 \right)$$

Exercise:

Problem 4.3 Rybicky & Lightman p 149

Show that in the instantaneous rest frame of a particle [Use the Lorentz transformations. Not dimensional arguments !!]

$$a_{\parallel}' = \gamma^3 a_{\parallel}$$
$$a_{\perp}' = \gamma^2 a_{\perp}$$

Dimensional Analysis:

One easy way to understand these transformations is to recall that acceleration is the second time derivative of a length increment

$$a_{\parallel}' \sim \frac{\Delta x_{\parallel}'}{(\Delta t')^2} \qquad\qquad \Delta x_{\parallel}' \sim \gamma\, \Delta x_{\parallel}$$

$$\sim \frac{\gamma\, \Delta x_{\parallel}}{(\frac{1}{\gamma}\, \Delta t)^2} \qquad\qquad \Delta t \sim \gamma\, \Delta t'$$

$$\sim \gamma^3 \frac{\Delta x_{\parallel}}{(\Delta t)^2}$$

$$\sim \gamma^3 a_{\parallel}$$

$$a_{\perp}' \sim \frac{\Delta x_{\perp}'}{(\Delta t')^2}$$

$$\sim \frac{\Delta x_{\perp}}{(\frac{\Delta t}{\gamma})^2} \qquad\qquad \Delta x_{\perp}' = \Delta x_{\perp}$$

$$\sim \gamma^2 \frac{\Delta x_{\perp}}{(\Delta t)^2}$$

$$\sim \gamma^2\, a_{\perp}$$

$$\Rightarrow \quad \left.\begin{aligned} a_{\parallel}' &= \gamma^3 a_{\parallel} \\ a_{\perp}' &= \gamma^2 a_{\perp} \end{aligned}\right\} \text{ In instantaneous rest frame } K'$$

The generalization of the Larmor equation is then

$$
\begin{aligned}
P_e &= P_e' \\
&= \frac{2e^2}{3c^3}\left(a_{\parallel}'^2 + a_{\perp}'^2\right) \\
&= \frac{2e^2}{3c^3}\left(\gamma^6 a_{\parallel}^2 + \gamma^4 a_{\perp}^2\right) \\
&= \frac{2e^2}{3c^3}\gamma^4\left(\gamma^2 a_{\parallel}^2 + a_{\perp}^2\right)
\end{aligned}
$$

Dont be fooled by the $\gamma^2 a_{\parallel}^2$, this component is hardly important. Since the velocity of relativistic particles are always close to speed of light, very little parallel acceleration is achieved - remember the effect of relativistic mass !!

$$M = \gamma m_0$$

$$M \to \infty \qquad \gamma \ggg 1$$

$$
\begin{aligned}
a_{\parallel} &= \frac{F_{\parallel}}{M} \\
&= \frac{F_{\parallel}}{\gamma m_0}
\end{aligned}
$$

For $F_{\parallel}$ fixed: $a_{\parallel} \Rightarrow 0$ for $\gamma \to \infty$

One can proceed to calculate the acceleration and hence the radiation from particles spiralling in magnetic fields. The resultant radiation is called Synchrotron radiation.

One assumption is that the synchrotron losses are not important during one revolution about a magnetic field line. Let's look at this process:

The relativistic Lorentz force in the absence of electric fields is:

$$\bar{F}_L = \frac{d}{dt}(\gamma m \bar{v})$$

$$= \frac{e}{c}(\bar{v} \times \bar{B}) = \frac{e}{c}\left[(\bar{v}_{\parallel} + \bar{v}_{\perp}) \times \bar{B}\right]$$

$\Rightarrow \bar{v}_{\parallel} \times \bar{B} = 0$

The parallel and perpendicular components of this force are

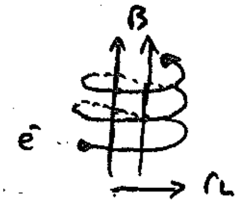
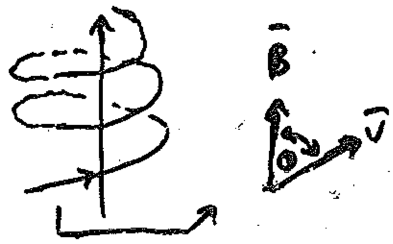


Fig 4.1

$$F_{L\parallel} \approx \frac{e}{c} v_{\parallel} B \sin\theta$$

$$\approx 0 \quad \text{for } [\theta \to 0^\circ \;\; \sin\theta \to 0 \;\; F_{\parallel} \to 0]$$

Hence $F_{L\parallel} \to 0$ for $\theta \to 0$



Therefore

$$F_{L\parallel} = \frac{d}{dt}(\gamma m_0 v_{\parallel})$$

$$= \gamma m_0 \frac{dv_{\parallel}}{dt}$$

$$\to 0$$

$$\Rightarrow F_{L\parallel} \to 0 \Rightarrow \frac{dv_{\parallel}}{dt} \to 0$$ [No acceleration along magnetic field]

The perpendicular component is:

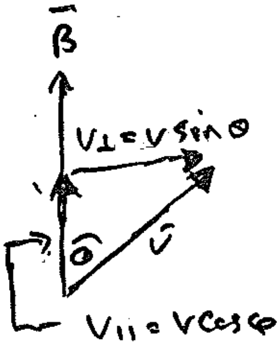
$$F_{L\perp} = \frac{d}{dt}(\gamma m_0 V_\perp)$$
$$= \frac{e}{c} V_\perp B \sin\theta$$
$$(F_{L\perp})_{max} = \frac{e V_\perp B}{c} \qquad \text{for } \theta = 90^\circ \quad \sin\theta \Rightarrow 1$$

Then

$$(F_{L\perp})_m = \frac{d}{dt}(\gamma m_0 V_\perp) = \frac{e V_\perp B}{c}$$

speed remains the same

$$\left[\gamma m_0 \frac{dV_\perp}{dt}\right] = \frac{e V_\perp B}{c}$$
$$a_\perp = \frac{e V_\perp B}{\gamma m_0 c}$$
$$a_\perp = \frac{e V \sin\theta B}{\gamma m_0 c}$$



The Larmor radius of the particle's orbit is

$$a_\perp = \frac{V_\perp^2}{r_L}$$
$$r_L = \frac{V_\perp^2}{a_\perp}$$
$$= \frac{V_\perp^2}{\left(\frac{e V_\perp B}{\gamma m_0 c}\right)}$$
$$= \frac{\gamma m_0 c V_\perp}{eB} \qquad V_\perp = V \sin\theta$$
$$= \frac{\gamma m_0 c V \sin\theta}{eB}$$
$$= \frac{\gamma m_0 c^2 \beta \sin\theta}{eB} \qquad \beta = \frac{v}{c}$$

The fundamental frequency is the inverse of the time required to complete one orbit (gyration frequency)

$$T_{gyr} = \frac{2\pi r_L}{v_\perp}$$

$$\nu_B = \frac{1}{T_{gyr}}$$

$$= \frac{v_\perp}{2\pi r_L}$$

$$= \frac{v \sin\theta}{2\pi r_L}$$

$$= \frac{v \sin\theta}{2\pi \left[\frac{\gamma m_0 c^2 \beta \sin\theta}{eB} \right]}$$

$$= \frac{eB}{2\pi \gamma m_0 c}$$

$$= \frac{1}{\gamma} \left[\frac{eB}{2\pi m_0 c} \right]$$

$$= \frac{1}{\gamma} \nu_L \qquad \nu_L = \frac{eB}{2\pi m_0 c}$$

gyration frequency of non-relativistic particle.

$$\nu_B \propto \frac{B}{\gamma}$$

$\nu_B \propto B$ Large $B \Rightarrow$ Large magnetic Lorentz force making particle gyrates

$\nu_B \propto \frac{1}{\gamma}$ Large $\gamma \Rightarrow$ large particle inertia and hence large r_L and smaller frequency.

The dominant contribution to synchrotron power mentioned earlier is

$$P_s = \frac{2e^2}{3c^3} \gamma^4 a_\perp^2 \qquad [a_\parallel \Rightarrow 0]$$

$$= \frac{2e^2}{3c^3} \gamma^4 \left[\frac{evB\sin\theta}{\gamma m_0 c} \right]^2$$

$$= \frac{2e^4}{3m_0^2c^3} \gamma^2 \beta^2 B^2 \sin^2\theta$$

We can cast this in a more more convenient form by noticing that

1.) $U_{mag} = \frac{B^2}{8\pi}$ [magnetic energy density]

2.) The quantity $\left(\frac{e^2}{m_0c^2}\right)$ is the classical electron radius

3.) The square of the electron radius is proportional to

$$\sigma_T = \frac{8\pi}{3} r_0^2 \qquad r_0^2 = \left(\frac{e^2}{m_0c^2}\right)^2$$
$$= 6.65 \times 10^{-25}\ \text{cm}^2$$

Therefore

$$P_s(\theta) = \frac{2e^4}{3m_0^2c^3} \left(\frac{B^2}{8\pi}\right) 8\pi\, \gamma^2 \beta^2 \sin^2\theta$$

$$= \frac{16\pi c}{3} \left(\frac{e^2}{m_0c^2}\right)^2 \gamma^2\beta^2 \left(\frac{B^2}{8\pi}\right) \sin^2\theta$$

$$= 2c \left[\frac{8\pi}{3}\left(\frac{e^2}{m_0c^2}\right)^2\right] \left[\frac{B^2}{8\pi}\right] \gamma^2\beta^2\sin^2\theta$$

$$= 2c\, \sigma_T U_B \gamma^2 \beta^2 \sin^2\theta$$

In the case of an isotropic distribution of pitch angles we can take the average $\langle \sin^2\theta \rangle$ over all space.

Exercise: Show that the average over all space of

$$\langle \sin^2\theta \rangle = \frac{2}{3}$$

Hint:

$$\langle \sin^2\theta \rangle = \frac{\int_{\Omega} \sin^2\theta \, d\Omega}{\int_{\Omega} d\Omega}$$

with

$$d\Omega = \sin\theta \, d\theta \, d\phi$$

Therefore

$$\langle P(\theta) \rangle = 2c\, \sigma_T U_B \gamma^2 \beta^2 \langle \sin^2\theta \rangle$$
$$= \frac{4c}{3} \sigma_T U_B \gamma^2 \beta^2$$

Questions:

1.) Why is there a scattering cross section featuring in this eqn?

2.) What happens when $\theta \to 0$. Will the radiation stop or not? Explain!!

3.) What will you observe if the B-lines point to you and particles are rushing along B-field lines with velocities $\beta \to 1$. Hint: Remember that

$$\Rightarrow \sin\theta \sim \frac{1}{\gamma}$$

Synchrotron Cooling Time

When you want to determine the time scale for any quantity, you can get an order of magnitude estimate by

$$t = \frac{A}{\dot{A}}$$

The time it takes an electron to get rid of its energy through synchrotron radiation

$$t_{syn} = \frac{E}{\langle P_{syn} \rangle}$$

$$= \frac{\gamma m_o c^2}{\frac{4}{3} \sigma_T c \, U_B \gamma^2 \beta^2}$$

$$= \frac{\gamma m_o c^2}{\frac{4}{3} \sigma_T c \left(\frac{B^2}{8\pi}\right) \gamma^2 \beta^2}$$

$$= \frac{8\pi m_o c^2}{\frac{4}{3} \sigma_T c \, B^2 \gamma \beta^2}$$

$$\approx 7.7 \times 10^8 \frac{1}{\gamma B^2} \quad \text{sec}$$

$$\approx \frac{24.5}{\gamma B^2} \quad \text{yr}$$

For AGN we have, close to BH:

$\Rightarrow \gamma \sim 10^3$ and $B \approx 10^3$ Gauss

$$t_{syn} \approx 2.4 \times 10^{-8} \text{ yr} \left(\frac{\gamma}{10^3}\right)^{-1} \left(\frac{B}{10^3 \text{Gauss}}\right)^{-2}$$

$$\approx 0.8 \left(\frac{\gamma}{10^3}\right)^{-1} \left(\frac{B}{10^3 \text{Gauss}}\right)^{-2} \quad \text{sec}$$

For the radio lobes of certain radio-loud quasars the Lorentz factor

$$\gamma \sim 10^3$$

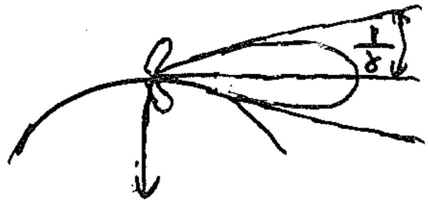
$$B \sim 10^{-5} \text{ Gauss}$$

$$t_{syn} = 245 \text{ million years} \left(\frac{\gamma}{10^3}\right)^{-1} \left(\frac{B}{10^{-5}\text{ Gauss}}\right)^{-2}$$

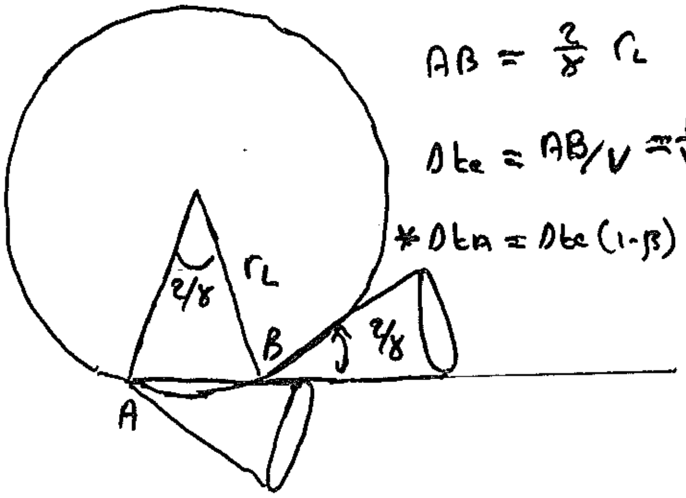
Spectrum Emitted by single particle

For relativistic electrons the frequency associated with the radiation is associated with the timescale during each revolution the radiation cone points in the direction of the observer. Remember the radiation cone

$$\sin\theta \sim \frac{1}{\gamma}$$



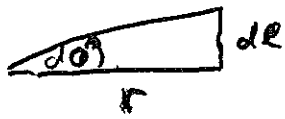
So the time during each orbit the observer receives radiation is, which will be associated with a certain frequency is (see Fig 4.3, Ghisellini)



$$AB = \frac{2}{\gamma} r_L$$

$$\Delta t_e = AB/v = \frac{1}{v}\left(r_L \frac{2}{\gamma}\right)$$

$$* \Delta t_A = \Delta t_e (1-\beta) \sim \Delta t_e \left(\frac{1}{2\gamma^2}\right)$$



$$dl = r\, d\theta$$

* We showed earlier that

$$\Delta t_{obs} = \Delta t_A = \left(1 - \frac{v}{c}\cos\theta\right)\Delta t_e$$

$$= \left(1 - \frac{v}{c}\right)\Delta t_e$$

{Here θ represents the direction of motion of e^- relative to obs.

The electron will beam radiation towards observer for a time

**

$$\boxed{\begin{aligned}(1-\beta) &= \frac{(1-\beta)(1+\beta)}{1+\beta} \\ &= \frac{(1-\beta^2)}{1+\beta} \\ &\approx \frac{1}{2\gamma^2}\end{aligned}}$$

$$\Delta t_e = \frac{AB}{v} \qquad \underbrace{AB}$$

$$= \frac{1}{v} \times \frac{2}{\gamma} r_L$$

$$= \frac{1}{v} \times \left[\frac{\gamma m_0 c^2 \beta \sin\theta}{eB}\right] \times \frac{2}{\gamma}$$

$$= \frac{2 m_0 c v}{v e B} \approx \frac{1}{\pi}\left(\frac{2\pi m_0 c}{eB}\right)$$

$$\approx \frac{1}{\pi}\frac{1}{\nu_L} \approx \frac{2}{e\pi}\left(\frac{1}{\gamma \nu_B}\right)$$

The observer will measure an interval between the arrival of the pulses emitted at A and B. Using our relations derived earlier:

$$\Delta t_A = \frac{\Delta t_e'}{\gamma_D}$$

$$= \gamma(1-\beta\cos\theta)\left(\frac{\Delta t_e}{\gamma}\right) \qquad (\Gamma = \gamma)$$

$$= (1-\beta\cos\theta)\,\Delta t_e$$

$$= (1-\beta)\,\Delta t_e \qquad \langle\theta\rangle \to 0 \text{ between } AB$$

$$= \frac{(1-\beta)(1+\beta)}{1+\beta}\,\Delta t_e$$

$$= \frac{(1-\beta^2)}{1+\beta}\,\Delta t_e$$

$$= \frac{1}{\gamma^2}\left(\frac{1}{1+\beta}\right)\Delta t_e$$

$$\approx \frac{1}{2\gamma^2}\,\Delta t_e \qquad \beta \to 1$$

$$= \frac{1}{2\gamma^2}\left[\frac{2}{2\pi}\ \frac{1}{\gamma\nu_B}\right]$$

$$\approx \left(\frac{1}{2\pi\gamma^3\nu_B}\right)$$

Remark:
This result also follows directly from:

$$\Delta t_A \approx \Delta t_e\left(\frac{1}{2\gamma^2}\right)$$

$$\approx \frac{2}{2\pi}\left(\frac{1}{\gamma\nu_B}\right)\left(\frac{1}{2\gamma^2}\right)$$

$$= \frac{2}{4\pi}\ \frac{1}{\gamma^3\nu_B}$$

$$= \frac{1}{2\pi\gamma^3\nu_B}$$

The inverse of this time is the typical synchrotron angular frequency $\omega = 2\pi\nu_s$

$$\omega = 2\pi\nu_s \approx \frac{1}{\Delta t_A}$$

$$\Rightarrow \nu_s = \frac{1}{2\pi\,\Delta t_A}$$

$$\approx \frac{1}{2\pi}\ \frac{1}{\left(\frac{1}{2\pi\gamma^3\nu_B}\right)}$$

$$\approx \gamma^3\nu_B \approx \gamma^3\left(\frac{\nu_L}{\gamma}\right) \approx \gamma^2\nu_L$$

$$\approx \gamma^2\left(\frac{eB}{2\pi m_e c}\right)$$

The Synchrotron Spectrum for single e^-

A more accurate description : R+L 79. Chapt 6

Main Idea:

The power per unit frequency emitted by an electron with a given Lorentz factor

$$P_s(\nu, \gamma, \theta) = \frac{\sqrt{3}\, e^3 B \sin\theta}{m_e c^2} F\left(\frac{\nu}{\nu_c}\right)$$

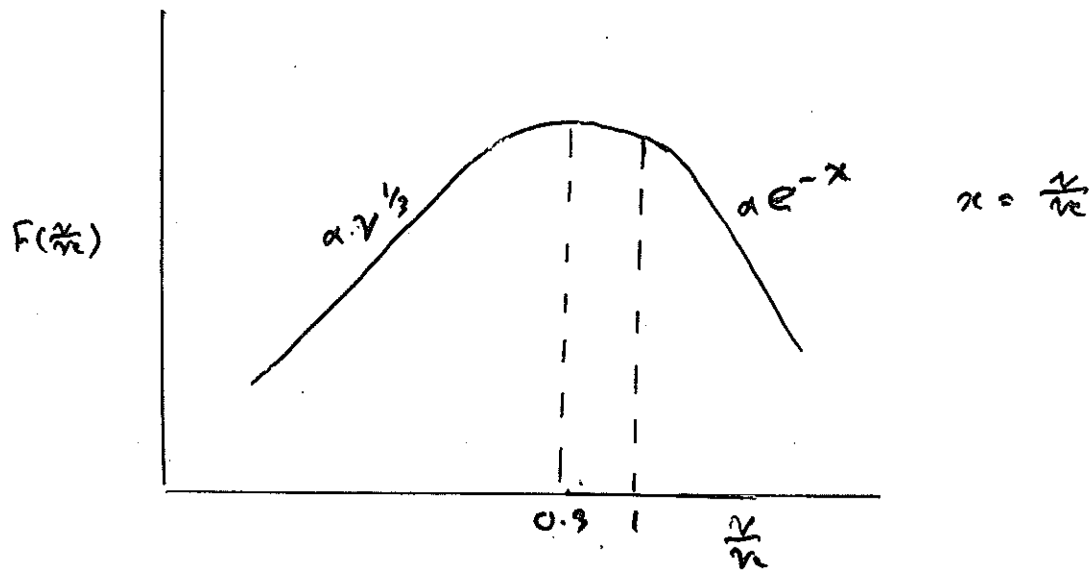
with

$$F\left(\frac{\nu}{\nu_c}\right) = \frac{\nu}{\nu_c} \int_{\nu/\nu_c}^{\infty} K_{5/3}(y)\, dy$$

$$\nu_c = \frac{3}{2} \nu_s \sin\theta$$

$$\nu_s = \frac{1}{2\pi \Delta t_A} = \gamma^3 \nu_B = \gamma^2 \nu_L = \gamma^2 \left(\frac{eB}{2\pi m_e c}\right)$$

The spectral behaviour is contained in



The function $F(\frac{\nu}{\nu_c})$ peaks at $\nu \sim 0.29\,\nu_c$ and have the following asymptotic limits

$$\nu \ll \nu_c \quad : \quad F(\tfrac{\nu}{\nu_c}) \to \frac{4\pi}{\sqrt{3}\,\Gamma(\frac{1}{3})} \left(\frac{\nu}{2\nu_c}\right)^{1/3}$$

$$\nu \gg \nu_c \quad : \quad F(\tfrac{\nu}{\nu_c}) \to \left(\frac{\pi}{2}\right)^{1/2} \left(\frac{\nu}{\nu_c}\right)^{1/2} e^{-\frac{\nu}{\nu_c}}$$

An approximation valid for most frequencies is

$$F(\tfrac{\nu}{\nu_c}) \sim \frac{4\pi}{\sqrt{3}\,\Gamma(\frac{1}{3})} \left(\frac{\nu}{2\nu_c}\right)^{1/3} e^{-\frac{\nu}{\nu_c}}$$

Limit of validity

One has to remember that during one orbit of the e^- around the B-field the emitted energy is much less than the e^- energy. If not the radiation will carry away so much energy that it will quickly modify the orbit and our calculations will not be valid. For non-relativistic particles this translates to

$$h\nu_B < m_0 c^2 \quad \text{[rest mass energy dominates KE]}$$

$$h\left(\frac{eB}{2\pi\gamma m_0 c}\right) < m_0 c^2$$

$$B < \underbrace{\frac{2\pi m_0^2 c^3}{he}}_{\text{just physical constants.}} \equiv B_c \quad (\gamma \to 1)$$

$$B \lesssim 4.4 \times 10^{13} \text{ Gauss}$$

$$\Rightarrow \quad B_c = \frac{2\pi m_e^2 c^3}{he}$$

This is the critical B-field above which quantum effects start to set in.

For relativistic particles we demand that the energy emitted during one orbit does not exceed the relativistic energy of the particles

$$\frac{P_s}{\nu_B} < \gamma m_0 c^2$$

$$P_s = \frac{2e^4}{3 m_e^2 c^3} B^2 \gamma^2 \beta^2 \sin^2\theta$$

$$\nu_B = \frac{eB}{2\pi \gamma m_0 c}$$

$$\Rightarrow \frac{\frac{2e^4}{3 m_e^2 c^3} B^2 \gamma^2 \beta^2 \sin^2\theta}{\left(\frac{eB}{2\pi\gamma m_0 c}\right)} < \gamma m_0 c^2$$

This gives

$$B < \frac{3 e m_e^2 c^3}{4\pi e^4 \gamma^3 \sin^2\theta \, m_0 c} \times \gamma m_0 c^2$$

$$< \frac{e}{\sigma_T} \cdot \frac{1}{\gamma^2 \sin^2\theta}$$

$$< \frac{7.2 \times 10^{14}}{\gamma^2 \sin^2\theta} \text{ Gauss}$$

Notice

$$\sigma_T = \frac{8\pi}{3} \left(\frac{e^2}{m_0 c^2}\right)^2$$

The Emission from many electrons

The most probable particle distribution in high energy astrophysics is the power-law

$$N(\gamma) = K\gamma^{-p} \quad [m^{-3}\ \gamma^{-1}]$$

$$\Rightarrow \quad N(\gamma) = \frac{dN}{d\gamma} \quad \Rightarrow N(\gamma)\,d\gamma = \frac{dN}{d\gamma}\,d\gamma$$

$$= K\gamma^{-p}\,d\gamma$$

Notice that

$$N(\gamma)\,d\gamma = \frac{dN}{d\gamma}\,d\gamma$$

$$= \frac{dN}{d\varepsilon}\frac{d\varepsilon}{d\gamma}\,d\gamma$$

$$= K_1\,\varepsilon^{-p}\frac{d\varepsilon}{d\gamma}\,d\gamma$$

$$= (m_0c^2 K_1)\,\varepsilon^{-p}\,d\gamma$$

$$= (m_0c^2 K_1)\,\varepsilon^{-p}\,\frac{d\varepsilon}{m_0c^2}$$

$$= K_1\,\varepsilon^{-p}\,d\varepsilon$$

$$= N(\varepsilon)\,d\varepsilon$$

$$\varepsilon = \gamma m_0c^2$$

$$\gamma = \frac{\varepsilon}{m_0c^2}$$

$$\boxed{d\gamma = \frac{d\varepsilon}{m_0c^2}}$$

$$\boxed{\frac{d\varepsilon}{d\gamma} = m_0c^2}$$

with $\quad N(\varepsilon) = \dfrac{dN}{d\varepsilon} \quad [m^{-3}\ \mathrm{erg}^{-1}]$

$$= K_1\,\varepsilon^{-p}$$

$$K = [m^{-3}\ \gamma^{p-1}]$$

$$K_1 = [m^{-3}\ \mathrm{erg}^{p-1}]$$

If we assume the same pitch angle distribution for low and high energy particles we can calculate the emissivity for the plasma

The emission function of the plasma in this case is called the emissivity [$\varepsilon_s(\nu,\theta) = j_\nu$ (erg cm^{-3} s^{-1} Hz^{-1} sr^{-1})]
Therefore

$$\varepsilon_s(\nu,\theta) = \frac{1}{4\pi}\int_{\gamma_{min}}^{\gamma_{max}} N(\gamma)\, P(\gamma,\nu,\theta)\, d\gamma$$

where
$$P(\gamma,\nu,\theta) = \frac{\sqrt{3}\, e^3 B \sin\theta}{m_0 c^2} \quad F\left(\frac{\nu}{\nu_c}\right)$$

The emissivity is therefore

$$\varepsilon_s(\nu,\theta) = \frac{1}{4\pi}\int_{\gamma_{min}}^{\gamma_{max}} k\gamma^{-p}\, \frac{\sqrt{3}\, e^3 B \sin\theta}{m_0 c^2} \quad F\left(\frac{\nu}{\nu_c}\right) d\gamma$$

One should keep in mind that

$$\nu_c = \frac{3}{2}\nu_s \sin\theta$$
$$= \frac{3}{2}\gamma^2 \frac{eB}{2\pi m_0 c} \sin\theta$$

$$\frac{\nu_c}{\nu} = \frac{3}{2}\,\frac{\gamma^2}{\nu}\,\frac{eB}{2\pi m_0 c} \sin\theta$$

$$\gamma^2 = \frac{2}{3}\left(\frac{\nu}{\nu_c}\right)^{-1} \nu \,\frac{2\pi m_0 c}{eB} \sin^{-1}\theta$$

$$\Rightarrow \quad \gamma = \sqrt{\frac{2}{3}}\left(\frac{\nu}{\nu_c}\right)^{-1/2} \nu^{1/2} \left(\frac{2\pi m_0 c}{eB}\right)^{1/2} \sin^{-1/2}\theta$$
$$= \left(\frac{\nu}{\nu_c}\right)^{-1/2} \nu^{1/2} \left(\frac{4\pi m_0 c}{3eB}\right)^{1/2} \sin^{-1/2}\theta$$
$$\Rightarrow \quad \gamma^{-p} = \left(\frac{\nu}{\nu_c}\right)^{+p/2} \nu^{-p/2} \left(\frac{4\pi m_0 c}{3eB}\right)^{-\frac{p}{2}} \sin^{p/2}\theta$$

Then also from

$$\gamma = \left(\frac{\nu}{\nu_c}\right)^{-1/2} \nu^{1/2} \left(\frac{4\pi m_e c}{3eB}\right)^{1/2} \sin^{-1/2}\theta$$

$$\Rightarrow \quad d\gamma = \nu^{1/2}\left(\frac{4\pi m_e c}{3eB}\right)^{1/2} \sin^{-1/2}\theta \; \nu^{1/2} \; d\left(\frac{\nu}{\nu_c}\right)^{-1/2}$$

$$|d\gamma| = \frac{1}{2}\nu^{1/2}\left(\frac{4\pi m_e c}{3eB}\right)^{1/2} \sin^{-1/2}\theta \; \nu^{1/2} \; \frac{1}{2}\left(\frac{\nu}{\nu_c}\right)^{-3/2} d\left(\frac{\nu}{\nu_c}\right)$$

Remember that $\nu_c = \nu_c(\gamma)$ and that the spectral properties are contained in

$$F\left(\frac{\nu}{\nu_c}\right)$$

hence the new integration variable

$$\Rightarrow \quad d\gamma \rightarrow d\left(\frac{\nu}{\nu_c}\right)$$

The emissivity is then

$$\varepsilon_s(\nu,\theta) = \frac{1}{4\pi}\int_{\gamma_{min}}^{\gamma_{max}} N(\gamma)\, P(\gamma,\nu,\theta)\, d\gamma$$

$$= \frac{1}{4\pi}\int_{\gamma_{min}}^{\gamma_{max}} K\gamma^{-p} \times \left[\frac{\sqrt{3}\, e^3 B \sin\theta}{m_e c}\right] F\left(\frac{\nu}{\nu_c}\right) [d\gamma]$$

$$= \frac{K}{4\pi}\int_{\gamma_{min}}^{\gamma_{max}} \nu^{-p/2}\left[\frac{4\pi m_e c}{3eB}\right]^{-p/2} \sin^{p/2}\theta \left(\frac{\nu}{\nu_c}\right)^{p/2} \times$$

$$\left[\frac{\sqrt{3}\, e^3 B \sin\theta}{m_e c}\right] F\left(\frac{\nu}{\nu_c}\right) \times \left[\frac{1}{2}\nu^{1/2}\left(\frac{4\pi m_e c}{3eB\sin\theta}\right)^{1/2}\left(\frac{\nu}{\nu_c}\right)^{-3/2} d\left(\frac{\nu}{\nu_c}\right)\right]$$

$$= \frac{K}{4\pi}\left(\frac{4\pi m_e c}{3eB}\right)^{-\left(\frac{p-1}{2}\right)} \sin\theta^{\left(\frac{p+1}{2}\right)} \nu^{-\left(\frac{p-1}{2}\right)} \left(\frac{\sqrt{3}\, e^3 B}{m_e c}\right) \times$$

$$\int_{\gamma_{min}}^{\gamma_{max}} F\left(\frac{\nu}{\nu_c}\right)\left(\frac{\nu}{\nu_c}\right)^{\left(\frac{p-3}{2}\right)} d\left(\frac{\nu}{\nu_c}\right)$$

$$= \frac{K}{4\pi}\left(\frac{4\pi m_e c}{3e}\right)^{-\left(\frac{p-1}{2}\right)} \sin^{\frac{p+1}{2}}\theta \left(\frac{\sqrt{3}\, e^3}{m_e c}\right) \times \left[\int_{\gamma_{min}}^{\gamma_{max}} F\left(\frac{\nu}{\nu_c}\right)\left(\frac{\nu}{\nu_c}\right)^{\left(\frac{p-3}{2}\right)} d\left(\frac{\nu}{\nu_c}\right)\right] \times B^{\left(\frac{p+1}{2}\right)} \nu^{-\left(\frac{p-1}{2}\right)}$$

Therefore

$$\mathcal{E}_s(\nu,\alpha) = \text{Const}\; B^{\left(\frac{p+1}{2}\right)}\; \nu^{-\left(\frac{p-1}{2}\right)}$$

$$= \text{Const}\; B^{\left(\frac{p+1}{2}\right)}\; \nu^{-\alpha}$$

$$\alpha = \left(\frac{p-1}{2}\right)$$

Notice that :

$$N(\gamma) \propto \gamma^{-p}$$

$$\mathcal{E}_s \propto \nu^{-\left(\frac{p-1}{2}\right)}$$

$$\propto \nu^{-\alpha}$$

This result can also be derived from first principles. To do this we must realize that the synchrotron spectrum emitted by a single particle is peaked. Most of the power is emitted around

$$\nu_s \approx \gamma^2 \nu_L \qquad \nu_L = \frac{eB}{2\pi m_e c}$$

In other words there is a connection between the energy of the electron and the frequency of photon it radiates. To simplify the analysis we assume the pitch angle $\theta \sim 90°$. The differential emissivity within a frequency interval $d\nu$ is

$$\mathcal{E}_s(\nu)\, d\nu = \frac{1}{4\pi} P_s\; N(\gamma)\, d\gamma$$

But

$$\nu_s = \nu = \gamma^2 \nu_L$$

$$\gamma^2 = \left(\frac{\nu}{\nu_L}\right)$$

$$\gamma = \left(\frac{\nu}{\nu_L}\right)^{1/2} \Rightarrow \frac{d\gamma}{d\nu} = \frac{1}{2}\left(\frac{\nu}{\nu_L}\right)^{-1/2}\left(\frac{1}{\nu_L}\right)$$

$$= \frac{1}{2}\,\frac{\nu^{-1/2}}{\nu_L} \times \nu_L^{1/2}$$

$$= \frac{1}{2}\left(\frac{\nu^{-1/2}}{\nu_L^{1/2}}\right)$$

Hence

$$\varepsilon_s(\nu) = \frac{1}{4\pi} P_s\, N(\gamma)\, \frac{d\gamma}{d\nu} \qquad P_s \propto \gamma^2 B^2$$

$$\varepsilon_s(\nu) \sim \frac{1}{4\pi}\left(\gamma^2 B^2\right) K \gamma^{-p}\, \frac{1}{2}\left(\frac{\nu^{-1/2}}{\nu_L^{1/2}}\right)$$

$$\sim \left(\frac{\nu}{\nu_L}\right)^2 B^2 \left(\frac{\nu}{\nu_L}\right)^{-p}\left(\frac{\nu}{\nu_L}\right)^{-1/2} \nu_L^{-1/2}\, \nu_L^{-1/2}$$

$$\sim \nu_L^{-1} B^2 \left(\frac{\nu}{\nu_L}\right)^{-\left(\frac{p-1}{2}\right)}$$

$$\sim \nu_L^{-1}\, \nu_L^{\left(\frac{p-1}{2}\right)} B^2 \nu^{-\left(\frac{p-1}{2}\right)}$$

$$\sim \nu_L^{\left(\frac{p-3}{2}\right)} B^2 \nu^{-\left(\frac{p-1}{2}\right)}$$

Since

$$\nu_L \propto B$$

$$\varepsilon_s(\nu) \sim B^{\left(\frac{p-3}{2}\right)} B^2 \nu^{-\left(\frac{p-1}{2}\right)}$$

$$\sim B^{\left(\frac{p+1}{2}\right)} \nu^{-\left(\frac{p-1}{2}\right)}$$

The synchrotron flux measured from a homogeneous and thin source with volume $V \sim R^3$ at a distance d_L is determined as follows:

$$F_S(\nu) \approx \frac{L_S(\nu)}{4\pi d_L^2}$$

$$\approx \frac{4\pi\, \varepsilon_S(\nu).\, V}{4\pi d_L^2}$$

$$\approx 4\pi\, \varepsilon_S(\nu) \times \frac{V}{4\pi d_L^2}$$

$$\approx \varepsilon_S(\nu) \times \frac{V}{4\pi d_L^2}$$

$$\propto B^{\left(\frac{p+1}{2}\right)}\, \nu^{-\left(\frac{p-1}{2}\right)} \times \frac{R^3}{d_L^2}$$

$$\propto \left(\frac{R}{d_L}\right)^2 R\, \nu^{-\alpha}\, B^{1+\alpha}$$

Here we used $\alpha = \frac{p-1}{2} \Rightarrow p = 1 + 2\alpha$

$$\Rightarrow F_S(\nu) \propto \Theta_S^2 R\, B^{1+\alpha}\, \nu^{-\alpha} \quad \begin{array}{l}(\text{watt m}^{-2}\,\text{Hz}^{-1}) \\ (\text{erg cm}^{-2}\,\text{s}^{-1}\,\text{Hz}^{-1})\end{array}$$

By observing the source at two frequencies we can deduce the spectral slope α. This allow us to determine the slope of the particle spectrum

$$N(\gamma) \sim \gamma^{-p} \qquad \alpha = \left(\frac{p-1}{2}\right)$$

Synchrotron Absorption

All emission processes have their own absorption counterpart. For synchrotron radiation the emission is done by relativistic electrons and they dont have a pure Maxwellian distribution. We could then have expected the same behaviour as for an optically thick BB

$$I_\nu \propto \nu^2 T$$

But for relativistic electrons inside the plasma we expect an equilibrium between the energy of the thermal particles ($\sim kT$) and the population of relativistic electrons ($\sim \gamma m_0 c^2$).

$$\Rightarrow kT \sim \gamma m_0 c^2 \qquad \nu = \gamma^2 \nu_L$$

$$\sim \left(\frac{\nu}{\nu_L}\right)^{1/2} m_0 c^2 \qquad \gamma^2 = \left(\frac{\nu}{\nu_L}\right)$$

$$\gamma = \left(\frac{\nu}{\nu_L}\right)^{1/2}$$

We showed earlier that predominantly photons for which $h\nu \ll kT$ are absorbed. In this limit

$$I_\nu = 2kT_b \frac{\nu^2}{c^2}$$

$$= 2\left[\gamma m_e c^2\right] \frac{\nu^2}{c^2}$$

$$= 2\left[\left(\frac{\nu}{\nu_L}\right)^{\frac{1}{2}} m_e c^2\right] \frac{\nu^2}{c^2}$$

$$= \frac{2 m_e c^2}{c^2} \nu_L^{-1/2} \nu^{5/2} \qquad [\nu_L \propto B]$$

$$\propto B^{-1/2} \nu^{5/2} \qquad [\mathrm{Watt\ m^{-2}\ Hz^{-1}\ sr^{-1}}]$$

or

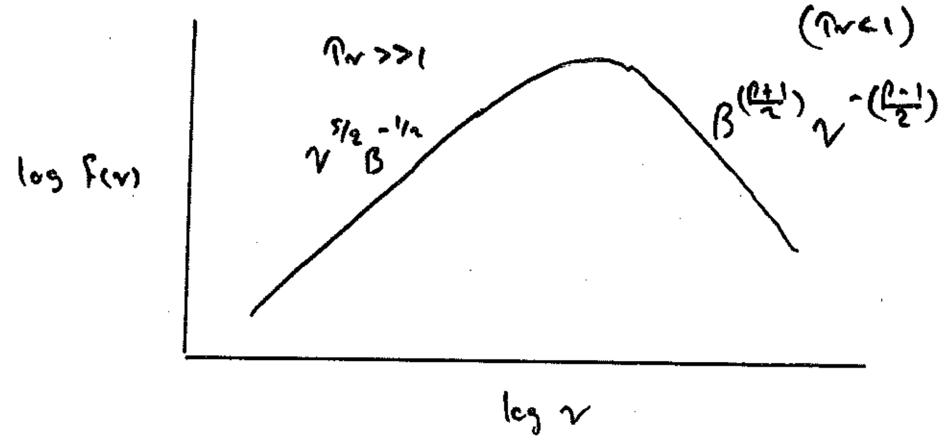
$$[\mathrm{erg\ cm^{-2}\ s^{-1}\ Hz^{-1}\ sr^{-1}}]$$

So the typical synchrotron spectrum is

$h\nu \ll kT$: $F_s(\nu) \approx I_\nu \Theta_s^2$

$\propto B^{-1/2} \Theta_s^2 \nu^{5/2}$

$h\nu \gg kT$: $F_s(\nu) \propto \Theta_s^2 R B^{\frac{p+1}{2}} \nu^{-\left(\frac{p-1}{2}\right)}$ $\alpha = \left(\frac{p-1}{2}\right)$



Observations of a synchrotron source at frequencies where the source is optically thick can allow us to constrain the magnetic field.

$$F_\nu^{thick} \propto B^{-1/2} \Theta_s^2 \nu^{5/2}$$

$$\Rightarrow B^{-1/2} \propto \frac{F_\nu^{thick}}{\Theta_s^2 \nu^{5/2}}$$

$$\Rightarrow B^{1/2} \propto \frac{\Theta_s^2 \nu^{5/2}}{F_\nu^{thick}}$$

$$\Rightarrow B \propto \frac{\Theta_s^4 \nu^5}{(F_\nu^{thick})^2}$$

From thick to thin : Turnover frequency

To describe the transition from self-absorbed ($\tau_\nu > 1$) to optically thin ($\tau_\nu < 1$), we make use of the radiative transfer equation

$$I_\nu = \frac{j_\nu}{\alpha_\nu}\left(1 - e^{-\tau_\nu}\right)$$
$$= \frac{\varepsilon(\nu)}{k_\nu}\left(1 - e^{-\tau_\nu}\right)$$

Remember : $\left.\begin{array}{l}\boxed{\varepsilon(\nu) = j_\nu} \\ \boxed{K_\nu = \alpha_\nu}\end{array}\right\}$ Ghisellini's notation

Also $\tau_\nu = K_\nu R \Rightarrow K_\nu = \dfrac{\tau_\nu}{R}$

$$I_\nu = \frac{\varepsilon(\nu)}{k_\nu}\left(1 - e^{-\tau_\nu}\right)$$
$$= \frac{\varepsilon(\nu) R}{k_\nu R}\left(1 - e^{-\tau_\nu}\right)$$
$$= \frac{\varepsilon(\nu) R}{\tau_\nu}\left(1 - e^{-\tau_\nu}\right)$$

For $\tau_\nu \gg 1$:

$$I_\nu = \varepsilon(\nu)\frac{R}{\tau_\nu} \qquad \left(e^{-\tau_\nu} \to 0\right)$$
$$= \varepsilon(\nu)\,\Delta R \qquad \left[\Delta R = \frac{R}{\tau_\nu}\right]$$

Emission comes from shell of thickness ΔR

Therefore

$$I_\nu = \frac{\varepsilon(\nu)}{\left(\frac{\tau_\nu}{R}\right)} = \frac{\varepsilon(\nu)}{k_\nu} \qquad \tau_\nu \gg 1$$

The absorption coefficient is for $\tau_\nu \gg 1$:

$$k_\nu = \frac{\varepsilon(\nu)}{I_\nu}$$

$$\propto \frac{K\, B^{\left(\frac{p+1}{2}\right)}\, \nu^{-\left(\frac{p-1}{2}\right)}}{\nu^{5/2}\, B^{-1/2}}$$

$$\propto K\, B^{\left(\frac{p+1}{2}\right)}\, B^{1/2}\, \nu^{-\left(\frac{p-1}{2}\right)}\, \nu^{-5/2}$$

$$\propto K\, B^{\left(\frac{p+2}{2}\right)}\, \nu^{-\left(\frac{p+4}{2}\right)} \qquad (cm^{-1})$$

The absorption coefficient is depending strongly on frequency. One can see that, as expected, at large frequency the absorption is negligible.

The transition between optically thick ($\tau_\nu > 1$) and thin ($\tau_\nu < 1$) occurs if $\tau_\nu = 1$. This will define the so-called turnover frequency, i.e ν_t.

$$\tau_{\nu t} = R\, k_{\nu t} = 1$$

$$R\, K\, B^{\left(\frac{p+2}{2}\right)}\, \nu_t^{-\left(\frac{p+4}{2}\right)} = 1$$

$$\nu_t^{-\left(\frac{p+4}{2}\right)} \propto \frac{B^{-\left(\frac{p+2}{2}\right)}}{R\, K}$$

$$\nu_t \propto \left[R\, K\, B^{\left(\frac{p+2}{2}\right)}\right]^{\frac{2}{p+4}}$$